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ON LINE GRAPHS WITH CROSSING NUMBER 2

D. G. Akka, S. Jendroľ, M. Klešč, S. V. Panshetty

Kulli, Akka and Beineke [5] established a characterization of planar graphs whose line graphs have crossing number 1. In [1], the same characterization was presented in terms of forbidden subgraphs. The main result of this paper is a characterization of planar graphs whose line graphs have crossing number 2.

1. INTRODUCTION

A graph is planar if it can be drawn in the plane in such a way that no two of its edges intersect. The *crossing number* $cr(G)$ of G is the least number of intersections of pairs of edges in any embedding of G in the plane. Obviously G is planar if and only if $cr(G) = 0$. It is implicit that the edges in a drawing are Jordan arcs (hence, nonselfintersecting), and it is easy to see that a drawing with minimum number of crossings (an optimal drawing) must be a *good* drawing; that is, each two edges have at most one point in common, which is either a common end vertex or a crossing. For other definitions see [**4**].

All graphs considered here are finite, undirected and without loops or multiple edges. The following theorems will be useful in the proof of our main theorem.

Theorem A. (see [**6**]) *The line graph of a planar graph G is planar if and only if $\Delta(G) \leq 4$ and every vertex of degree 4 is a cut vertex.*

We may revise Theorem A to read:

Theorem B. *The line graph of a planar graph G has crossing number 0 if and only if $\Delta(G) \leq 4$ and every vertex of degree 4 is a cut vertex.*

Theorem C. (see [**5**]) *The line graph of a planar graph G has crossing number 1 if and only if (1) or (2) holds:*

(1) $\Delta(G) = 4$ *and there is a unique not cut vertex of degree 4.*

(2) $\Delta(G) = 5$, *every vertex of degree 4 is a cut vertex, there is a unique vertex of degree 5 and it is a cut vertex having at most 3 incident edges in any block.*

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2. MAIN RESULT

Theorem. *The line graph $L(G)$ of a planar graph G has crossing number 2 if and only if one of the following conditions holds*:

(1) $\Delta(G) = 4$ *and exactly two of the vertices of degree* 4 *are not cut vertices of* G.

(2) $\Delta(G) = 5$, *there are exactly two vertices of degree* 5, *each is a cut vertex of* G, *and each has at most* 3 *incident edges in any block. Every vertex of degree* 4 *is a cut vertex.*

(3) $\Delta(G) = 5$, *there is a unique vertex of degree* 5, *it is a cut vertex having at most* 3 *incident edges in any block, and there is a unique not cut vertex of degree* 4 *in* G.

(4) $\Delta(G) = 5$, *there is a unique vertex of degree* 5, *it is a cut vertex having exactly* 4 *incident edges in one block, and, moreover, either at least one of the four vertices adjacent to the vertex of degree* 5 *in the block has degree* 2 *or in the block there is a vertex of degree* 2, *which together with the vertex of degree* 5 *forms a cut set of the block. Every vertex of degree* 4 *is a cut vertex of* G.

3. THREE LEMMAS

First we prove a three lemmas which are applied in the proof of our Theorem.

Lemma 1. *If in G there is a vertex of degree* 5 *which is not a cut vertex of* G, *than* $L(G)$ *has at least* 3 *crossings.*

Proof. Assume $\deg(v) = 5$ and v is not a cut vertex of G. The edges incident with the vertex v enforce in $L(G)$ a complete graph on five vertices which we denote by K_5^v. It is known (see for example [**2**]) that every good drawing of K_5 has an odd number of crossings and that $cr(K_5) = 1$. Let D be a good drawing of $L(G)$. If in D the edges of K_5^v cross each other (the internal crossings of K_5^v) at least three times, we are done.

Suppose that K_5^v has exactly one internal crossing in D. Then the subdrawing D^* of D induced by the vertices of K_5^v creates the map with 8 regions, because the optimal drawing of K_5 is unique within isomorphism (the crossing is considered to be a vertex of the map). Since in G any two edges incident with v are on a cycle, any two vertices of K_5^v are in $L(G)$ on a cycle containining only one edge of K_5^v. One can easy to see that such cycles containing the edges of K_5^v which cross each other have with D^* two more crossings. Thus, in D, there are at least three crossings. □

Lemma 2. *Let v be a cut vertex of degree* 5 *in G and let four edges incident with v be in one block B of G. If in B all vertices adjacent to v have degrees at least* 3 *and there is no vertex u of degree* 2 *such that the vertex set* $\{u, v\}$ *forms a cut set of* B, *then* $L(G)$ *has at least three crossings.*

Proof. By hypothesis, the subgraph $B - v$ of G contains no cut vertex of degree 2, and this implies that its line graph $L(B - v)$ does not contain a bridge. Let D be a good drawing of $L(G)$. If, in D, the subgraph K_5^v has exactly one internal crossing, then the subdrawing D^* of K_5^v in D induces the map. In the case when K_5^v has more than 1 internal crossing it has at least three ones.

First suppose that the subgraph $L(B-v)$ of $L(G)$ lie in more than one region of D^*. Since $L(B - v)$ does not contain a bridge, its edges cross the edges of D^* at least twice and so there are at least three crossings in D.

Now suppose that in D the subgraph $L(B - v)$ lie in one region of D^*. By hypothesis, every vertex of B adjacent to v has degree at least 3. Therefore, every of four vertices of K_5^v belonging to $L(B)$ is adjacent with at least two vertices of $L(B - v)$. Since in D^* there are at most three vertices of K_5^v on the boundary of every region there are, in D, at least two crossings between the edges of K_5^v and the edges joining vertices of $L(B - v)$ and K_5^v. This completes the proof. □

Lemma 3. *Let G' be a graph obtained from G by the transformation shown in Figure 1, where v is a vertex of degree 4 which is not a cut vertex of G. If $1 \leq cr(L(G)) < 3$, then $cr(L(G')) < cr(L(G))$.*

Proof. Let $\deg(v) = 4$ and let v is not a cut vertex of G. The edges incident with the vertex v form in $L(G)$ the complete graph on four vertices which we denote by K_4^v. We note that in every good drawing of $L(G)$ at least one of the edges of K_4^v is crossed, otherwise a contraction of the edges of $L(G) - K_4^v$ into one vertex results a graph isomorphic to K_5, but without crossings.

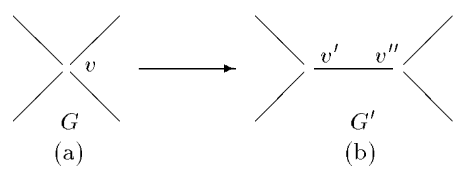

Figure 1

Let D be an optimal drawing of $L(G)$ with fewer than 3 crossings. Then the subdrawing D^{**} obtained from D by deleting all edges of K_4^v has all vertices of K_4^v on the boundary of one region. Otherwise, in D, the edges of K_4^v are crossed at least three times. Now we can draw into this region of D^{**} one vertex and six edges (the line graph of the subgraph of G' as in Figure 1 (b), see Figure 2 (a)) without crossing to obtain a drawing of $L(G')$. Since, in D, at least one of the edges of K_4^v is crossed, and the drawing D is optimal, $cr(L(G')) < cr(L(G))$. □

4. PROOF OF MAIN THEOREM

Suppose that the line graph $L(G)$ of a planar graph G has crossing number 2. Then, by Theorem B, we have $\Delta(G) \geq 4$.

First we assume that $\Delta(G) = 4$. It follows from Theorems B and C, that G has at least two not cut vertices of degree 4. Suppose that G has three not cut vertices of degree 4. Applying Lemma 3 to one of these vertices we can obtain G' with two not cut vertices of degree 4 whose line graph has fewer than two crossings. This contradicts Theorem C. Thus, G has exactly two not cut vertices of degree 4.

Assume $\Delta(G) = 5$. By Lemma 1, every vertex of degree 5 is a cut vertex. G has at most two vertices of degree 5, otherwise $L(G)$ contains at least three subgraphs isomorphic to K_5, each with at least one crossing among its edges. Suppose G has two cut vertices u and v of degree 5, and let u has four incident edges in one block. Theorem C implies that both u and v are in the same block of G. Without loss of generality we may assume that they are not adjacent, because by inserting a vertex of degree 2 between u and v we obtain a graph whose line graph has no more crossings than $L(G)$. In every good drawing of $L(G)$ there is at least one crossing among the edges of K_5^v. Thus, by contracting the edges of K_5^v into one vertex we obtain a line graph of a graph containing u with four incident edges in one block. This line graph has crossing number at most one, which contradicts Theorem C. Therefore both u and v have at most three incident edges in a block. Moreover, using Lemma 3 and Theorem C, one can easy to see that every vertex of degree 4 is a cut vertex.

Suppose now there is a unique vertex v of degree 5 which is a cut vertex of G. If v has at most three incident edges in one block then, by Theorem C, there is, in G, at least one not cut vertex of degree 4. By Lemma 3 and Theorem C, it is easy to show that in this case there is a unique vertex of degree 4 that is not a cut vertex of G.

Let v be a unique cut vertex of degree 5 in G and it has exactly four incident edges in one block. By Lemma 3 and Theorem C, every vertex of degree 4 is a cut vertex of G. Moreover, at least one vertex adjacent to v in the block with 4 edges incident with v has degree 2 or in that block there is a vertex of degree 2 which together with v form a cut set of the block. Otherwise, by Lemma 2, $L(G)$ has at least 3 crossings.

Finally assume $\Delta(G) \geq 6$ and let $\deg(v) = n \geq 6$. Then $L(G)$ contains a subgraph K_6 with at least 3 crossings among its edges (see for example [3]). This is a contradiction.

Conversely, assume G satisfies the given conditions; then by Theorem C, $L(G)$ has crossing number at least 2. If (1) holds, then v_1 and v_2 (adjacent or nonadjacent) are two not cut vertices of degree 4. Using transformation from Figure 1 on both vertices v_1 and v_2 one can obtain G''. Then, by Theorem A, $L(G'')$ is planar. This can be transformed to give a drawing of $L(G)$ with two crossings (see Figure 2).

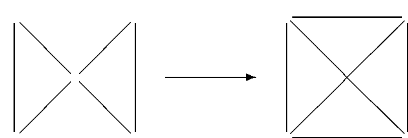

Figure 2

Now assume that the condition (2) holds. Let v_1 and v_2 (adjacent or nonadjacent) be two vertices of degree 5. Then the edges incident with v_1 can be split into two sets of sizes 2 and 3 in such a way that no edges in different sets are in the same block. Form G' from G by the transformation as in Figure 3(a). Then, by Theorem C, $cr(L(G')) = 1$, e_6 is a cut vertex of $L(G')$ and the vertices of the block of $L(G^{'})$ containing the vertices e_3, e_4, e_5 and e_6, but other than these vertices, lie in the region with e_3, e_4 and e_5 on its boundary. We can assume that the vertices of the block of $L(G^{'})$ containing the edge $\{e_1, e_2\}$ other than e_1, e_2 and e_6 lie in the triangular region with the vertices e_1, e_2 and e_6 on its boundary. The transformation of $L(G^{'})$ into $L(G)$ with exactly two crossings is shown in Figure 3(b).

Next suppose that the condition (3) holds. The edges incident with the vertex v of degree 5 can be split into two sets of sizes 2 and 3 so that no edges in different sets are in the same block. Transform G to $G^{'}$ as in Figure 3(a). Then $\Delta(G^{'}) = 4$ and $G^{'}$ contains one not cut vertex u of degree 4. By Theorem C, $cr(L(G')) = 1$ and the line graph of the block containing u is, in $L(G^{'})$ (see Figure 3(a)), either in the triangular region with e_1, e_2 and e_6 on its boundary or in the region with e_3, e_4 and e_5 on its boundary. This can be again transformed to obtain a drawing of $L(G)$ with one additional crossing as shown in Figure 3(b).

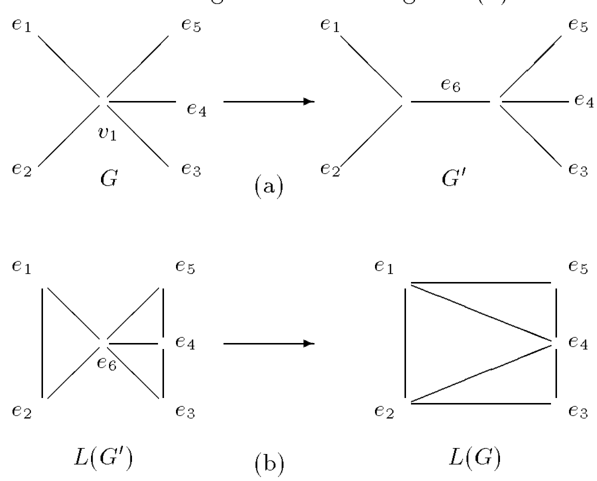

Figure 3

Finally, suppose the condition (4) holds. Let u and v be vertices of degree 2 and degree 5, respectively, mentioned in the condition (4). Let e_1, e_2, e_3, e_4 and e_5 be edges incident with the vertex v such that e_1 is a bridge and the other edges belong to a subgraph of G_2, where G_2 is a connected subgraph of G not containing e_1. Let G_1 be a subgraph of G induced by edges of G not belonging to G_2. By Theorem C, $cr(L(G_2)) = 1$ and $L(G_1)$ is planar. Because $L(G_2 - u)$ is planar (see Theorem B), the graph $L(G_2)$ can be drawn in such a way that the edges of the subgraph K_4 of $L(G_2)$ induced by the edges e_2, e_3, e_4 and e_5 do not cross one another and one crossing of $L(G_2)$ is realized with one of these edges and the edge (in fact K_2) which associates, in $L(G)$, to the vertex u. Let us draw $L(G_1)$ (of course without crossings) into a triangular region of K_4, not containing inside any vertex of $L(G_2)$, in such a way that the vertex e_1 is on the outer face with respect to the drawing of $L(G_1)$. Then we can join the vertex e_1 with the vertices e_2, e_3, e_4

and e_5 of $L(G_2)$ not producing more than one crossing. The result is a drawing of $L(G)$ having exactly two crossings. This completes the proof.

REFERENCES

1. D. G. Akka, S. V. Panshetty: *Forbidden subgraphs for graphs with line graphs of crossing numbers* ≤ 1. Periodica Mathematica Hungarica **26** (1993), 175–185.
2. D. Archdeacon, R. B. Richter: *On the parity of crossing numbers.* J. Graph Theory **12** (1988), 307–310.
3. R. K. Guy: *Latest results on crossing numbers.* In: Recent Trends in Graph Theory, Springer N. Y. (1971), 143–156.
4. F. Harary: *Graph Theory.* Addison – Wesley Readings (1969).
5. V. R. Kulli, D. G. Akka, L. W. Beineke: *On line graphs with crossing number* 1. J. Graph Theory **3** (1979), 87–90.
6. J. Sedláček: *Some properties of interchange graphs.* Theory of Graphs and Its Applications (M. Fiedler, ed.), Academic Press, New York (1962), 145–150.

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Mathematics Department,
V. G. Women's College Gulbarga,
Karnataka,
India

Department of Geometry and Algebra,
P. J. Šafárik University, Košice,
and Slovak Academy of Sciences,
Jesenná 5, 041 54 Košice,
Slovak Republic

Department of Mathematics,
Faculty of Electrical Engineering and Informatics,
Technical University, Košice,
Letná 9, 042 00 Košice,
Slovak Republic

Mathematics Department,
B. V. B. College, Bidar,
Karnataka,
India