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PYTHAGOREAN THEOREM IN UNITARY SPACES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Many properties of right triangles have been generalized to unitary spaces of dimension n, sometimes in disguise. In this article, a generalization of Pythagoras' Theorem related to areas and volumes is presented.

1. NOTATION

E_n denotes an n-dimensional unitary space and $\mathbf{R}_n$ an n-dimensional Euclidean space. Vectors are denoted by greek letters $\alpha, \beta, \ldots$, and scalars by latin letters. The inner product of ξ and η will be (ξ, η), and the norm of ξ is defined by $\|\xi\| = (\xi, \xi)^{1/2}$. $\vec{0}$ indicates the zero vector. Well-known ideas of linear spaces will be assumed. Other definitions will be presented as needed.

2. RIGHT PYRAMIDS IN $\mathbf{R}_3$

A right pyramid in $\mathbf{R}_3$ is a tetrahedron for which all three angles at one vertex are right angles. The face opposite this vertex is called the surface-hypotenuse. In the language of vectors, one can define this as follows.

Let $\{\xi, \eta, \zeta\}$ be a set of non-zero orthogonal vectors in $\mathbf{R}_3$. The convex hull of $\{\vec{0}, \xi, \eta, \zeta\}$ is a right pyramid whose vertex is at $\vec{0}$ (Figure 1). The triangle whose vertices are the endpoints of ξ, η, and ζ is called the *surface-hypotenuse.*

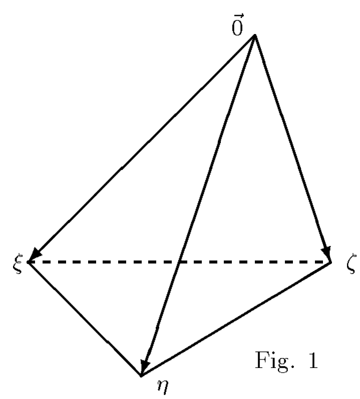

Fig. 1

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Theorem 1. *In a right pyramid the square of the area of the surface-hypotenuse is equal to the sum of the squares of the areas of the other faces.*
Proof. The sum of the squares of the three faces in Figure 1 is

$$\mathcal{A}^2 = \frac{1}{4}\left(||\xi||^2||\eta||^2 + ||\eta||^2||\zeta||^2 + ||\zeta||^2||\xi||^2\right). \tag{1}$$

The area of the surface-hypotenuse is one-half the area of the parallelogram formed by the vectors $\xi - \zeta$ and $\xi - \eta$. It is well-known that the square of this area is

$$\mathcal{B}^2 = \frac{1}{4}\det\begin{pmatrix} (\xi-\eta,\xi-\eta) & (\xi-\eta,\xi-\zeta) \\ (\xi-\eta,\xi-\zeta) & (\xi-\zeta,\xi-\zeta) \end{pmatrix}. \tag{2}$$

Since $(\xi,\eta) = (\xi,\zeta) = (\eta,\zeta) = 0$, this simplifies to

$$\mathcal{B}^2 = \frac{1}{4}\det\begin{pmatrix} ||\xi||^2 + ||\eta||^2 & ||\xi||^2 \\ ||\xi||^2 & ||\xi||^2 + ||\zeta||^2 \end{pmatrix}. \tag{3}$$

So

$$\mathcal{B}^2 = \frac{1}{4}\left(||\xi||^2||\eta||^2 + ||\eta||^2||\zeta||^2 + ||\zeta||^2||\xi||^2\right) = \mathcal{A}^2.$$

3. GENERALIZATION TO n DIMENSIONS

Although one can generalize Pythagoras' Theorem in many ways, the approach of Section 2 seems quite natural.

The right pyramid in $\mathbf{R}_n$ is the convex hull of $\{\vec{0}, \xi_1, \ldots, \xi_n\}$, where $\{\xi_1, \ldots, \xi_n\}$, is an orthogonal set of non-zero vectors in $\mathbf{R}_n$. The convex hull of the endpoints $\xi_1, \ldots, \xi_n$ is called the *hyperhypotenuse* of the right pyramid. The convex hull of $\{\vec{0}, \xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_n\}$, is caled a *hyperface* of the pyramid, where $\widehat{\xi_k}$, $k = 1, \ldots, n$ indicates that element is left out of the sequence.

Theorem 2. *Let $\xi_1, \ldots, \xi_n$ be an orthogonal set of vectore in $\mathbf{R}_n$ forming the edges of a right pyramid. Let $\mathcal{V}$ be the volume of the hyperhypotenuse of the right pyramid, and let $\mathcal{V}_k$ be the volume of the hyperface which is the convex hull of $\{\vec{0}, \xi_1, \ldots, \widehat{\xi_k}, \ldots, \xi_n\}$. Then*

$$\mathcal{V}^2 = \sum_{k=1}^{n} {\mathcal{V}_k}^2. \tag{4}$$

Proof. As before, we have

$$\sum_{k=1}^{n} {\mathcal{V}_k}^2 = \left(\frac{1}{(n-1)!}\right)^2 \sum_{k=1}^{n}\left(||\xi_1||^2 \cdots \widehat{||\xi_k||}^2 \cdots ||\xi_n||^2\right) \tag{5}$$

and

$$(6) \qquad \mathcal{V}^2 = \left(\frac{1}{(n-1)!}\right)^2 \det \begin{pmatrix} (\xi_1-\xi_2,\xi_1-\xi_2) & \dots & (\xi_1-\xi_2,\xi_1-\xi_n) \\ \vdots & & \vdots \\ (\xi_1-\xi_n,\xi_1-\xi_2) & \dots & (\xi_1-\xi_n,\xi_1-\xi_n) \end{pmatrix}$$

It will be important to note later that the determinant in equation (6) is unchanged under permutations of the edges $\xi_1,\dots,\xi_n$. This is clear geometrically because the area of the hyperhypotenuse does not depend on the order in which the edges are given.

By the orthogonality of $\xi_1,\dots,\xi_n$ equation (6) becomes

$$(7) \qquad \mathcal{V}^2 = \left(\frac{1}{(n-1)!}\right)^2 \det \begin{pmatrix} ||\xi_1||^2+||\xi_2||^2 & ||\xi_1||^2 & \dots & ||\xi_1||^2 \\ \vdots & & & \vdots \\ ||\xi_1||^2 & & \dots & ||\xi_1||^2+||\xi_n||^2 \end{pmatrix}.$$

Denoting $||\xi_1||^2,\dots,||\xi_n||^2$ repectively by $a_1,\dots,a_n$, this becomes

$$(8) \qquad \mathcal{V}^2 = \left(\frac{1}{(n-1)!}\right)^2 \det \begin{pmatrix} a_1+a_2 & a_1 & \dots & a_1 \\ a_1 & a_1+a_3 & \cdots & a_1 \\ \vdots & & & \vdots \\ a_1 & \dots & & a_1+a_n \end{pmatrix}.$$

Let $f(a_1,\dots,a_n)$ denote the determinant in equation (8). It will suffice to prove the identity

$$(9) \qquad f(a_1,\dots,a_n) = \sum_{k=1}^{n}(a_1a_2\cdots\widehat{a_k}\cdots a_n).$$

Although this could be proved by a direct but somewhat messy inductive argument, it will turn out to be much simpler to give an argument based on symmetry.

It is clear from its form that $f(a_1,\dots,a_n)$ is a polynomial of degree at most $n-1$ in the variables $a_1,\dots,a_n$. Since the determinant in equation (6) is unchanged under permutation of the edges $\xi_1,\xi_2,\dots,$ f is in fact a *symmetric polynomial*, i.e, $f(a_1,\dots,a_n) = f(a_{\sigma(1)},\dots,a_{\sigma(n)})$ for any permutation σ of $\{1,\dots,n\}$.

Symmetry will be used to calculate the terms of f. by Setting $a_1 = 0$, the terms not involving a_1 will be obtained:

$$(10) \qquad f(0,a_2,\dots,a_n) = \det \begin{pmatrix} a_2 & 0 & \dots & 0 \\ 0 & a_3 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & a_n \end{pmatrix} = a_2\cdots a_n = \widehat{a_1}a_2\cdots a_n.$$

For any term of a symmetric polynomial, the distinct terms obtained from it by a permutation of the variables are also present. So

$$(11) \qquad f(a_1,\dots,a_n) = \sum_{k=1}^{n}(a_1\cdots\widehat{a_k}\cdots a_n) + g(a_1,\dots,a_n),$$

where g, being the difference of two symmetric polynomials, is also a symmetric polynomial. Every non-zero term of g must contain a_1, since we have already found the only term of f that doesn't. By symmetry, every non-zero term of g must contain $a_2, \ldots, a_n$ as well. Thus, every non-zero term of g must have degree at least n, which is impossible. Hence, g is identically 0, and

$$f(a_1, \ldots, a_n) = \sum_{k=1}^{n} (a_1 \cdots \widehat{a_k} \cdots a_n) \tag{12}$$

as required.

An analogous theorem is true for $\xi_1, \ldots, \xi_n$ element of the unitary space E_n. The proof, which is nearly identical, is omitted.

Between the usual formulation of Pythagoras' Theorem

$$\left\| \sum_{k=1}^{n} \xi_k \right\|^2 = \sum_{k=1}^{n} ||\xi||^2 \tag{13}$$

and the generalization just given in Theorem 2 are a number of other equalities. Their formulations will be left to the reader.

4. ADJOINT PRODUCTS

Let A and B be linear transformations on E_n. Recall that the adjoint A^* is defined by $(A\xi, \eta) = (\xi, A^*\eta)$ for every $\xi, \eta \in E_n$ ([**1**], [**2**]).

A^*B is defined to be the *adjoint product* of A and B. If $A^*B = 0$, then A is said to be *adjoint orthogonal* to B. It is clear that $A^*B = 0$ implies $B^*A = 0$.

The theorem of Section 3 can be generalized to the Hilbert norm of a set of adjoint orthogonal linear transformations on E_n. An outline of this will be presented.

Definition 3. *Let $\{A_1, \ldots, A_n\}$ be a set of non-zero orthogonal linear transformations on E_n. The hyperhypotenuse of this set is defined by*

$$\mathcal{S} = \{A_1 - A_2, A_1 - A_3, \ldots, A_1 - A_n\}, \tag{14}$$

and the square of the norm of $\mathcal{S}$ is defined by

(15)
$$\mathcal{N} = \max_{||\xi||=1} \det \begin{pmatrix} ((A_1 - A_2)\xi, (A_1 - A_2)\xi) & \ldots & ((A_1 - A_2)\xi, (A_1 - A_n)\xi) \\ \vdots & & \vdots \\ ((A_1 - A_n)\xi, (A_1 - A_2)\xi) & \ldots & ((A_1 - A_n)\xi, (A_1 - A_n)\xi) \end{pmatrix}.$$

The hypersurfaces of the set are $\{A_1, \ldots, \widehat{A_k}, \ldots, A_n\}$. The square of the norm of each hypersurface is defined as

$$\mathcal{N}_k = \max_{||\xi||=1} \left\{ \frac{(A_1\xi, A_1\xi) \cdot (A_2\xi, A_2\xi) \cdots (A_n\xi, A_n\xi)}{(A_k\xi, A_k\xi)} \right\}, \quad k = 1, \ldots, n. \tag{16}$$

Lemma 4. *Let λ_k be the square of the Hilbert norm of A_k for $k = 1, \ldots, n$. Then*

$$\mathcal{N}_k = \frac{\lambda_1 \cdots \lambda_n}{\lambda_k}\,. \tag{17}$$

Theorem 5. *Let $\{A_1, \ldots, A_n\}$ be an adjoint orthogonal set of linear transformations on E_n. Then*

$$\mathcal{N} = (\lambda_1 \cdots \lambda_n) \sum_{i=1}^{n} \frac{1}{\lambda_i}\,. \tag{18}$$

Proof. It is observed that

$$\det \begin{pmatrix} \big((A_1 - A_2)\xi, (A_1 - A_2)\xi\big) & \ldots & \big((A_1 - A_2)\xi, (A_1 - A_n)\xi\big) \\ \vdots & & \vdots \\ \big((A_1 - A_n)\xi, (A_1 - A_2)\xi\big) & \ldots & \big((A_1 - A_n)\xi, (A_1 - A_n)\xi\big) \end{pmatrix} \tag{19}$$

$$= \det \begin{pmatrix} \|A_1\xi\|^2 + \|A_2\xi\|^2 & \ldots & \|A_1\xi\|^2 \\ \vdots & & \vdots \\ \|A_1\xi\|^2 & \ldots & \|A_1\xi\|^2 + \|A_n\xi\|^2 \end{pmatrix}.$$

Comparing equation (19) with equation (7) and applying the method of Theorem 2, the proof is easily completed.

Again, this result may be generalized in many ways.

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