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A GENERALIZATION OF THE KY FAN INEQUALITY

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

A certain extension of the Ky Fan inequality is proved by means of elementary calculus.

1. INTRODUCTION

The following inequality due to KY FAN was recorded in [**1**]:

$$\left(\frac{\prod\limits_{i=1}^{n} x_i}{\prod\limits_{i=1}^{n}(1-x_i)}\right)^{1/n} < \frac{\sum\limits_{i=1}^{n} x_i}{\sum\limits_{i=1}^{n}(1-x_i)} \qquad (0 \le x_i \le 1/2), \tag{1}$$

unless $x_1 = x_2 = \ldots = x_m$.

With the notation

$$M_p(x) = \left(\frac{1}{n}\sum_{i=1}^{n} {x_i}^p\right)^{1/p} \quad (x_i > 0) \;\text{ and }\; M_0(x) = \lim_{p\to 0} M_p(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n},$$

(1) becomes

$$\frac{M_0(x)}{M_0(1-x)} < \frac{M_1(x)}{M_1(1-x)}\,. \tag{2}$$

D. SEGAIMAN [**2**] conjectured that

$$\frac{M_p(x)}{M_p(1-x)} \le \frac{M_q(x)}{M_q(1-x)} \qquad (p \le q). \tag{3}$$

F. CHAN, D. GOLDBERG and S. GONEK [**2**] gave some counter examples when $0 < 2^p/p < 2^q/q$ or $p + q > 9$. In addition, they proved that (3) is true for $p + q = 0 > p$ or $0 \le p \le 1 \le q \le 2$.

Recently the case $p = -1$ and $q = 0$ was proved to be true by WAN–LAN WANG and PENG–FEI WANG [**3**]. And the case $-1 \le p \le 0 \le 1$ was proved

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by Guang–Xing Li and Ji Chen [**4**]. Zhen Wang and Ji Chen [**5**] proved that the function $R(p) = M_p(x)/M_p(1-x)$ is strictly increasing on $[-1,1]$, unless $x_1 = \ldots = x_n$, and if (3) holds for (p,q), then also for $(-q,-p)$.

In this paper, we determine all exponents p and q such that (3) is true.

Theorem. *For arbitrary n, $p<q$, the inequality*

$$\left(\frac{\sum_{i=1}^{n} {x_i}^p}{\sum_{i=1}^{n}(1-x_i)^p}\right)^{1/p} \le \left(\frac{\sum_{i=1}^{n} {x_i}^q}{\sum_{i=1}^{n}(1-x_i)^q}\right)^{1/q} \qquad (0 < x_i \le 1/2) \tag{4}$$

holds if and only if $|p+q| \le 3$, $2^p/p \ge 2^q/q$ when $p>0$, $p2^p \le q2^q$ when $q<0$.

The proof of the sufficiency is contained in Sections 3, 4 and 5. In the proof we assume $pq \ne 0$, otherwise by letting p or $q \to 0$, it is easy to see that (4) is also true. In Section 2, we will prove the necessity.

2. PROOF OF THE NECESSITY

In [**2**], it is proved that (4) and $p<q$ are equivalent when $n=2$, and that if (4) holds, then $2^p/p \ge 2^q/q$ for $p>0$.

When $q<0$, take $x_1 = x_2 = \cdots = x_{n-1} = \varepsilon$ $(0<\varepsilon<1/2)$ and $x_n = 1/2$. Then (4) becomes

$$\left(\frac{(n-1)\varepsilon^p + (1/2)^p}{(n-1)(1-\varepsilon)^p + (1/2)^p}\right)^{1/p} \le \left(\frac{(n-1)\varepsilon^q + (1/2)^q}{(n-1)(1-\varepsilon)^q + (1/2)^q}\right)^{1/q}, \tag{5}$$

or

$$\frac{\left(\varepsilon^p + \frac{1}{2^p(n-1)}\right)^{1/p}}{\left(\varepsilon^q + \frac{1}{2^q(n-1)}\right)^{1/q}} \le \frac{\left((1-\varepsilon)^p + \frac{1}{2^p(n-1)}\right)^{1/p}}{\left((1-\varepsilon)^q + \frac{1}{2^q(n-1)}\right)^{1/q}}. \tag{6}$$

Letting $\varepsilon \to 0$, (6) yields

$$1 \le \frac{\left(1 + \frac{1}{2^p(n-1)}\right)^{1/p}}{\left(1 + \frac{1}{2^q(n-1)}\right)^{1/q}}, \tag{7}$$

hence

$$\left(1 + \frac{1}{2^p(n-1)}\right)^{1/p} \ge \left(1 + \frac{1}{2^q(n-1)}\right)^{1/q}. \tag{8}$$

By using the Maclaurin expansion in $1/n$, we obtain

$$1 + (p2^pn)^{-1} + o(1/n^2) \ge 1 + (q2^qn)^{-1} + o(1/n^2). \tag{9}$$

So if $p\,2^p > q\,2^q$, (4) would be false for sufficiently large n.

In the equivalent inequality of (4):

$$\left(\frac{\sum_{i=1}^{n}(1-u_i)^p}{\sum_{i=1}^{n}(1+u_i)^p}\right)^{1/p} \le \left(\frac{\sum_{i=1}^{n}(1-u_i)^q}{\sum_{i=1}^{n}(1+u_i)^q}\right)^{1/q} \qquad (0 \le u_i < 1), \tag{10}$$

let $u_1 = u_2 = \cdots = u_{n-1} = 0$ and $u_n = u$ $(0 < u < 1)$, then (10) becomes

$$\left(\frac{(n-1)+(1-u)^p}{(n-1)+(1+u)^p}\right)^{1/p} \le \left(\frac{(n-1)+(1-u)^q}{(n-1)+(1+u)^q}\right)^{1/q}. \tag{11}$$

Take the Maclaurin expansion of (11) in u:

$$1 - \frac{2}{n}\,u + \frac{2}{n^2}\,u^2 - \frac{(n-1)\big((n-2)p^2 - 3np\big) + 2(n^2+2)}{3n^3}\,u^3 + o(u^4) \tag{12}$$

$$\le 1 - \frac{2}{n}\,u + \frac{2}{n^2}\,u^2 - \frac{(n-1)\big((n-2)q^2 - 3nq\big) + 2(n^2+2)}{3n^3}\,u^3 + o(u^4).$$

Thus for u sufficiently small, (10) holds only if

$$(n-2)p^2 - 3np \ge (n-2)q^2 - 3nq, \tag{13}$$

or

$$(p-q)\big((n-2)(p+q) - 3n\big) \ge 0. \tag{14}$$

So for $n \ge 3$, we have

$$p + q \le \frac{3n}{n-2}. \tag{15}$$

Letting $n \to +\infty$, (15) yields $p + q \le 3$.

Similarly, the expansion of (10) with $u_1 = u_2 = \cdots = u_{n-1} = u$ $(0 < u < 1)$, $u_n = 0$ gives

$$p + q \ge \frac{-3n}{n-2}. \tag{16}$$

So we obtain $p + q \ge -3$.

3. AN EQUIVALENCE PROPOSITION

In this section, we are to establish an equivalence proposition as follows:

Proposition. *For $p < q$, the following inequalities are equivalent*:

$$\left(\frac{\sum_{i=1}^{n}\lambda_i {x_i}^p}{\sum_{i=1}^{n}\lambda_i(1-x_i)^p}\right)^{1/p} < \left(\frac{\sum_{i=1}^{n}\lambda_i {x_i}^q}{\sum_{i=1}^{n}\lambda_i(1-x_i)^q}\right)^{1/q}, \tag{i}$$

where $\lambda_i > 0,\ 0 < x_i \le 1/2,\ i = 1, 2, \ldots, n$ *and* $x_1, x_2, \ldots, x_n$ *are not all equal;*

$$\text{(ii)} \qquad \left(\frac{\lambda x^p + \mu y^p}{\lambda(1-x)^p + \mu(1-y)^p}\right)^{1/p} < \left(\frac{\lambda x^q + \mu y^q}{\lambda(1-x)^q + \mu(1-y)^q}\right)^{1/q},$$

where $\lambda, \mu > 0,\ 0 < x \ne y \le 1/2;$

$$\text{(iii)} \qquad \left(\frac{\lambda + (1-u)^p}{\lambda + (1+u)^p}\right)^{1/p} < \left(\frac{\lambda + (1-u)^q}{\lambda + (1+u)^q}\right)^{1/q},$$

where $\lambda > 0,\ 0 < u < 1.$

Proof. (i) obviously implies (iii).

Now suppose (iii) is true, let $x > y$ and $y/x = 1 - u$, $x/(1-x) = k$, then $0 < u < 1,\ 0 < k \le 1$ and $(1-y)/(1-x) = 1 + ku$. So (ii) is equivalent to the following:

$$f(k) = \frac{1}{q}\ln\frac{\lambda + \mu(1-u)^q}{\lambda + \mu(1+ku)^q} - \frac{1}{p}\ln\frac{\lambda + \mu(1-u)^p}{\lambda + \mu(1+ku)^p} > 0. \tag{17}$$

Differentiating $f(k)$, one can obtain

$$\begin{aligned} f'(k) &= \frac{-\mu(1+ku)^{q-1}u}{\lambda + \mu(1+ku)^q} + \frac{\mu(1+ku)^{p-1}u}{\lambda + \mu(1+ku)^p} \\ &= \frac{u}{1+ku}\left(\frac{\mu(1+ku)^p}{\lambda + \mu(1+ku)^p} - \frac{\mu(1+ku)^q}{\lambda + \mu(1+ku)^q}\right) < 0. \end{aligned} \tag{18}$$

Hence

$$f(k) \ge f(1) = \frac{1}{q}\ln\frac{\lambda + (1-u)^q}{\lambda + (1+u)^q} - \frac{1}{p}\ln\frac{\lambda + (1-u)^p}{\lambda + (1+u)^p} > 0. \tag{19}$$

(ii) is established.

We will use induction to show that (i) is true if (ii) holds. At first, (ii) is the case $n = 2$ of (i). Now assume that (i) holds for some n $(n \ge 2)$.

Let $1/2 \ge x_1 \ge x_2 \ge \cdots \ge x_{n+1}$, and x_i are not all equal, then there exists $\mu > 0$ and $\nu = \lambda_1\lambda_{n+1}/\mu > 0$ such that

$$\begin{aligned} \frac{\sum_{i=1}^{n+1}\lambda_i {x_i}^p}{\sum_{i=1}^{n+1}\lambda_i(1-x_i)^p} &= \frac{\mu {x_1}^p + \lambda_{n+1}x_{n+1}^p}{\mu(1-x_1)^p + \lambda_{n+1}(1-x_{n+1})^p} \\ &= \frac{\lambda_1 {x_1}^p + \nu x_{n+1}^p}{\lambda_1(1-x_1)^p + \nu(1-x_{n+1})^p} = R^p. \end{aligned} \tag{20}$$

It is clear that $(\lambda_1 - \mu)(\lambda_{n+1} - \nu) \le 0$. Without loss of generality, we may assume that $\lambda_1 \geq \mu$. So

$$R^p = \frac{(\lambda_1 - \mu){x_1}^p + \sum\limits_{i=2}^{n} \lambda_i {x_i}^p}{(\lambda_1 - \mu)(1 - x_1)^p + \sum\limits_{i=2}^{n} \lambda_i (1 - x_i)^p} . \tag{21}$$

By the assumption, we have

$$R = \left(\frac{(\lambda_1 - \mu){x_1}^p + \sum\limits_{i=2}^{n} \lambda_i {x_i}^p}{(\lambda_1 - \mu)(1 - x_1)^p + \sum\limits_{i=2}^{n} \lambda_i (1 - x_i)^p} \right)^{1/p} \leq \left(\frac{(\lambda_1 - \mu){x_1}^q + \sum\limits_{i=2}^{n} \lambda_i {x_i}^q}{(\lambda_1 - \mu)(1 - x_1)^q + \sum\limits_{i=2}^{n} \lambda_i (1 - x_i)^q} \right)^{1/q} , \tag{22}$$

and

$$R = \left(\frac{\mu {x_1}^p + \lambda_{n+1} x_{n+1}^p}{\mu (1 - x_1)^p + \lambda_{n+1}(1 - x_{n+1})p} \right)^{1/p} < \left(\frac{\mu {x_1}^q + \lambda_{n+1} x_{n+1}^q}{\mu (1 - x_1)^q + \lambda_{n+1}(1 - x_{n+1})q} \right)^{1/q} . \tag{23}$$

So we have

$$R < \left(\frac{\sum\limits_{i=1}^{n+1} \lambda_i {x_i}^q}{\sum\limits_{i=1}^{n+1} \lambda_i (1 - x_i)^q} \right)^{1/q} . \tag{24}$$

Therefore, we get that (i) is true for arbitrary n and the proposition is established.

4. THREE LEMMAS

Lemma 1. *If* $\alpha \leq 0$, $\alpha < \beta \leq 1 - \alpha$, $0 \leq u < 1$, *then*

$$(1 + u)^\alpha + (1 - u)^\alpha \geq (1 + u)^\beta + (1 - u)^\beta . \tag{25}$$

The equality is attained if and only if $u = 0$ *or* $(\alpha, \beta) = (0, 1)$.
Proof. Let $\varphi(x) = (1 + u)^x + (1 - u)^x$ $(0 < u < 1)$, then

$$\varphi''(x) = (1 + u)^x \big(\ln (1 + u) \big)^2 (1 - u)^x \big(\ln (1 - u) \big)^2 > 0. \tag{26}$$

So we have to establish (25) only for $\beta = 1 - \alpha$, i.e.

$$\Phi(u) = (1 + u)^\alpha + (1 - u)^\alpha - \big((1 + u)^{1-\alpha} + (1 - u)^{1-\alpha} \big) > 0, \tag{27}$$

where $\alpha < 0,\ 0 < u < 1$.

$$\Phi(u) = 2\sum_{n=0}^{+\infty}\left(\binom{\alpha}{2n} - \binom{1-\alpha}{2n}\right)u^{2n} \tag{28}$$
$$= 2\alpha(\alpha-1)\sum_{n=2}^{+\infty}\frac{u^{2n}}{(2n)!}\left(\prod_{k=2}^{2n-1}(\alpha-k) - \prod_{k=2}^{2n-1}(-\alpha-k+1)\right)$$
$$\geq 2\alpha(\alpha-1)\sum_{n=2}^{+\infty}\frac{u^{2n}}{(2n)!}\left(\prod_{k=2}^{2n-1}(\alpha-k) - \prod_{k=2}^{2n-1}|-\alpha-k+1|\right)$$
$$> 0.$$

This proves the lemma.

Lemma 2. *If $0 < \alpha < \beta < 1-\alpha$ and $0 < u \leq 1$. Let*

$$G(u) = (1+u)^\alpha + (1-u)^\alpha - (1+u)^\beta - (1-u)^\beta, \tag{29}$$

then there exists an unique u_0, such that

(i) $G(u) > 0$ *for* $0 < u < u_0$;

(ii) $G(u) < 0$ *for* $u_0 < u \leq 1$.

Proof. We have $\beta(\beta-1) < \alpha(\alpha-1) < 0$, hence

$$0 < \frac{\alpha(\alpha-1}{\beta(\beta-1)} < 1. \tag{30}$$

Define

$$g(u) = \frac{(1+u)^{\beta-2} + (1-u)^{\beta-2}}{(1+u)^{\alpha-2} + (1-u)^{\alpha-2}} \qquad (0 < u < 1). \tag{31}$$

We have

$$g'(u) = \frac{(\beta-2)\big((1+u)^{\beta-3} - (1-u)^{\beta-3}\big)}{(1+u)^{\alpha-2} + (1-u)^{\alpha-2}} \tag{32}$$
$$- \frac{(\alpha-2)\big((1+u)^{\alpha-3} - (1-u)^{\alpha-3}\big)\big((1+u)^{\beta-2} + (1-u)^{\beta-2}\big)}{\big((1+u)^{\alpha-2} + (1-u)^{\alpha-2}\big)^2}$$
$$< \frac{\alpha-2}{\big((1+u)^{\alpha-2} + (1-u)^{\alpha-2}\big)^2}\Big(\big((1+u)^{\beta-3} - (1-u)^{\beta-3}\big) \times$$
$$\times \big((1+u)^{\alpha-2} + (1-u)^{\alpha-2}\big)$$
$$-\big((1+u)^{\alpha-3} - (1-u)^{\alpha-3}\big)\big((1+u)^{\beta-2} + (1-u)^{\beta-2}\big)\Big)$$
$$= \frac{2(\alpha-2)(1+u)^{\alpha+\beta-6}}{\big((1+u)^{\alpha-2} + (1-u)^{\alpha-2}\big)^2}\left(\left(\frac{1-u}{1+u}\right)^{\alpha-3} - \left(\frac{1-u}{1+u}\right)^{\beta-3}\right) < 0.$$

So $g(u)$ is strictly decreasing with $g(0) = 1$ and $g(1) = 0$. Hence there exists a unique $u_1 \in (0,1)$ such that

$$g(u_1) = \frac{\alpha(\alpha-1)}{\beta(\beta-1)}. \tag{33}$$

Note that

$$G'(u) = \alpha\big((1+u)^{\alpha-1} - (1-u)^{\alpha-1}\big) - \beta\big((1+u)^{\beta-1} - (1-u)^{\beta-1}\big), \tag{34}$$

$$G''(u) = -\beta(\beta-1)\big((1+u)^{\alpha-2} + (1-u)^{\alpha-2}\big)\left(g(u) - \frac{\alpha(\alpha-1)}{\beta(\beta-1)}\right), \tag{35}$$

and from above we know that $G''(u) > 0$ for $u \in (0, u_1)$, $G''(u) < 0$ for $u \in (u_1, 1)$.

Because $G(0) = 0$ and $G(1) = 2^\alpha - 2^\beta < 0$, we can find a unique $u_0 \in (u_1, 1)$ such that $G(u) > 0$ in $(0, u_0)$, $G(u) < 0$ in $(u_0, 1)$.

Lemma 3. *If $p < q$, $p + q \le 3$ and $2^p/p \ge 2^q/q$ for $p > 0$, then*

$$\frac{(1+u)^p - (1-u)^p}{p} \ge \frac{(1+u)^q - (1-u)^q}{q} \quad (0 \le u < 1), \tag{36}$$

equality occurs if and only if $u = 0$ or $(p,q) = (1,2)$.

Proof. Let

$$H(u) = \frac{(1+u)^p - (1-u)^p}{p} - \frac{(1+u)^q - (1-u)^q}{q} \quad (0 \le u < 1). \tag{37}$$

Then

$$H'(u) = \big((1+u)^{p-1} + (1-u)^{p-1}\big) - \big((1+u)^{q-1} + (1-u)^{q-1}\big). \tag{38}$$

When $p \le 1$, $p - 1 < q - 1 \le 1 - (p-1)$, by Lemma 1 we obtain $H'(u) \ge 0$. Thus $H(u) \ge H(0) = 0$ with equality if and only if $u = 0$ or $(p,q) = (1,2)$.

When $p > 1$, since the function $h(r) = 2^r/r$ strictly increases in $(0, \ln 2]$ and strictly decreases $[1/\ln 2, +\infty)$, and

$$h(1) = h(2), \quad h(p) \ge h(q) \qquad (p > 1), \tag{39}$$

we have $p + q < 1 + 2 = 3$, i.e. $0 < p - 1 < q - 1 < 1 - (p-1)$. By Lemma 2, $H'(u)$ has it's unique zero point u_0 in $(0,1)$, such that the following is true:

$H'(u) > 0$ for $0 < u < u_0$, hence $H(u) > H(0) = 0$;

$H'(u) < 0$ for $u_0 < u < 1$, hence $H(u) > H(1) = 2^p/p - 2^q/q \ge 0$;

and $H(u_0) > 0$.

These establish the lemma.

5. PROOF OF THE SUFFICIENCY

From the equivalence proposition in Section 3, we only need to prove the following inequality:

$$\left(\frac{\lambda+(1-u)^p}{\lambda+(1+u)^p}\right)^{1/p} < \left(\frac{\lambda+(1-u)^q}{\lambda+(1+u)^q}\right)^{1/q}, \tag{40}$$

where $\lambda > 0,\ 0 < u < 1,\ p < q,\ |p+q| \le 3,\ 2^p/p \ge 2^q/q$ when $p > 0$, $p\,2^p \le q\,2^q$ when $q < 0$.

The above inequality is equivalent to

$$F(\lambda) = \frac{1}{q}\ln\frac{\lambda+(1-u)^q}{\lambda+(1+u)^q} - \frac{1}{p}\ln\frac{\lambda+(1-u)^p}{\lambda+(1+u)^p} > 0. \tag{41}$$

But

$$\begin{aligned} F'(\lambda) &= \tfrac{1}{q}\left(\tfrac{1}{\lambda+(1-u)^q} - \tfrac{1}{\lambda+(1+u)^q}\right) - \tfrac{1}{p}\left(\tfrac{1}{\lambda+(1-u)^p} - \tfrac{1}{\lambda+(1+u)^p}\right) \\ &= (A\lambda^2 + B\lambda + C)/Q(\lambda), \end{aligned} \tag{42}$$

where

$$Q(\lambda) = \big(\lambda+(1-u)^q\big)\big(\lambda+(1+u)^q\big)\big(\lambda+(1-u)^p\big)\big(\lambda+(1+u)^p\big), \tag{43}$$

$$A = \frac{(1+u)^q-(1-u)^q}{q} - \frac{(1+u)^p-(1-u)^p}{p}, \tag{44}$$

$$\begin{aligned} B = {} & \big((1+u)^p+(1-u)^p\big)\,\frac{(1+u)^q-(1-u)^q}{q} \\ & - \big((1+u)^q+(1-u)^q\big)\,\frac{(1+u)^p-(1-u)^p}{p}, \end{aligned} \tag{45}$$

$$C = (1-u^2)^{p+q}\left(\frac{(1+u)^{-q}-(1-u)^{-q}}{-q} - \frac{(1+u)^{-p}-(1-u)^{-p}}{-p}\right). \tag{46}$$

By Lemma 3, when $(p,q) \ne (1,2)$ and $(p,q) \ne (-2,-1)$, we have $A < 0$ and $C > 0$. If $(p,q) = (1,2)$, then $A = 0$, $B = -4u^3 < 0$, $C = 2u^3(1-u^2) > 0$. If $(p,q) = (-2,-1)$, then $A = -2u^3/(1-u^2)^2 < 0$, $B = 4u^3/(1-u^2)^3 > 0$, $C = 0$. Thus for all these cases, $F'(\lambda)$ has an unique positive root λ_0 such that $F'(\lambda) > 0$ for $0 < \lambda < \lambda_0$; $F'(\lambda < 0$ for $\lambda > \lambda_0$. So

$$F(\lambda) > F(0) = F(+\infty) = 0 \ \text{ for } \ \lambda > 0. \tag{47}$$

Now the theorem is proved.

6. A CONJECTURE

Inequality (4) holds for all natural numbers n, and it is not the best result for each fixed n. We propose a conjecture for this condition as follows:

Conjecture. *If* $p < q$, $|p+q| \leq 3n/(n-2)$, $n \geq 3$,

$$(1 + 2^p/(n-1))^{1/p} \geq (1 + 2^q/(n-1))^{1/q} \quad \textit{when } p > 0, \tag{48}$$

and

$$(1 + 1/(2^p(n-1)))^{1/p} \geq (1 + 1/(2^q(n-1)))^{1/q} \quad \textit{when } q < 0, \tag{49}$$

then

$$\left(\frac{\sum_{i=1}^{n} {x_i}^p}{\sum_{i=1}^{n} (1-x_i)^p} \right)^{1/p} < \left(\frac{\sum_{i=1}^{n} {x_i}^q}{\sum_{i=1}^{n} (1-x_i)^q} \right)^{1/q} \qquad (0 < x_i \leq 1/2), \tag{50}$$

unless $x_1 = x_2 = \cdots = x_n$.

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