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A NOTE ON DIAMETERS OF MATROIDS BASES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

The concept of diameter of a matroid base is introduced to be the rank of the base complement. Since this diameter is related to the Hamming distance in the set of matroid bases, it will be also called Hamming diameter. It is proved (Theorem 1) that the set of diameters of all bases of a matroid satisfies the following property: If two positive integers belong to the set, then any integer between them also does. Two families of matroids (a graphic and a non-graphic one) with an arbitrarily wide range of base diameters are described.

1. INTRODUCTION

The diameter of a matroid base b is defined to be the rank of its complement. As it is shown (Lemma 1), diameter of b coincides with the maximum HAMMING distance (= cardinality of the set-difference) between b and any other base of the matroid. Therefore, diameter defined in this manner can be also called HAMMING diameter. The main result of this paper says that the set of diameters of all bases of a matroid satisfies the consecutive property in the sense that together with any pair of non-negative integers, this set contains all the integers in between (Theorem 1). Such a result has an important algorithmic implication. A construction of all matroid bases, together with their classification via diameters − may be iterated with decreasing diameters from the family of bases of maximum diameters and stopped at the first diameter with no corresponding base found.

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2. PRELIMINARIES

A *matroid* M on a finite ground-set E is an ordered pair (E, B), where B is an incommensurable family of special subsets of E, so-called *bases*, which satisfy the following *exchange operation axiom*:

If b_1 *and* b_2 *are two bases, and* $x \in b_2 \setminus b_1$ *then there exists an element* $y \in b_1 \setminus b_2$, *such that the set* $(b_1 \setminus \{y\}) \cup x$ *is also a base.*

It is said that the base $(b_1 \setminus \{y\}) \cup x$ is obtained from the base b_1 by an *exchange operation* . It is known that all the bases of a matroid have the same cardinality. Notice that HAMMING distance between two bases is equal to the minimal number of exchange operations needed to transform one of the bases into the other.

Given the family of bases of a matroid, one can easily define two related concepts:

An *independent set* of a matroid is any subset of a base. *Rank* of a subset X (denotation: $rank(X)$) of the ground-set E is the largest cardinality of an independent subset of X.

It is well-known (see, e.g. [**9**]) that the matroid rank is a non-decreasing function and that each independent set of a matroid can be augmented to a base.

Bases of a graphic matroid on a ground-set E coincide with spanning forests of an associated graph with the edge-set E.

In what follows, $|X|$ denotes the cardinality of a set X. Sets of numbers in examples will be written without brackets and commas.

3. DIAMETER OF A MATROID BASE

Throughout, b^* will denote the complement of a base b with respect to the ground-set. Given a base b of a matroid, its *diameter* is defined to be the rank of its complement, shortly: $diam(b) \stackrel{\text{def}}{=} rank(b^*)$. Accordingly, the family of bases of a matroid can be partitioned into classes with respect to diameter.

The notion "(Hamming) diameter" is justified by the following lemma:

Lemma 1. *Diameter of a base* b *of a matroid* M *is the largest Hamming distance between* b *and any other base of* M.

Proof. Each subset of a base is independent and its rank is equal to its cardinality. Besides, rank is a non-decreasing function. This implies that any base b' of M satisfies: $|b' \setminus b| = |b' \cap b^*| = rank(b' \cap b^*) \leq rank(b^*) = diam(b)$.

Next step is to prove that there exists a base with which the equality occurs. Let b_0 be a base of M that contains a maximal independent set I of b^*. Then $b_0 \setminus I$ is included into b and $|b_0 \setminus b| = |I| = rank(b^*) = diam(b)$. □

Given a matroid M, let $D(M)$ denote the set of diameters of all bases of M.

Theorem 1. *Given a matroid M, let $i, j \in D(M)$ so that $j - i \geq 2$ and let q be an integer satisfying $i < q < j$. Then $q \in D(M)$.*

Proof. Let b and b' be two bases of M such that $diam(b) - diam(b') \geq 2$. There exists a sequence $b = b_0, b_1, ..., b_k = b'$ of bases of M which is inductively generated from the pair (b, b'). The base b_{s+1} is generated by applying an exchange operation to the pair (b_s, b') for each s, $0 \leq s \leq k-1$, so that $|b' \setminus b_{s+1}| = |b' \setminus b_s| - 1$.

On the other hand, it holds that $diam(b_{s+1}) - diam(b_s) \in \{-1, 0, 1\}$ for $0 \leq s \leq k-1$. Namely, the applied exchange operation implies that $|b^*_{s+1} \setminus b^*_s| = |b^*_s \setminus b^*_{s+1}| = 1$, so both b^*_{s+1} and b^*_s arise by adding one element to $b^*_{s+1} \cap b^*_s$. Consequently, both $diam(b_{s+1}) = rank(b^*_{s+1})$ and $diam(b_s) = rank(b^*_s)$ belong to the set $\{rank(X), rank(X)+1\}$ and therefore these two diameters cannot differ by more than 1. This yields the required conclusion. □

Remarks.

1. Note that some of the bases in the sequence $b = b_0, b_1, ..., b_k = b'$ might have a diameter outside the range $[diam(b'), diam(b)]$.
2. Lemma 1 is a matroid version of the theorem, originally formulated and proved in the context of trees of a graph by Deo [**2**]. Theorem 1 is a matroid version of the theorem presented in [**7**] for graphs.
3. A theorem analogous to Theorem 1, for classical diameters of spanning trees of 2-connected graphs, is proposed in [**4**]. Notice that the Hamming diameter of a spanning tree T of a connected graph G is an entirely different notion from the classical diameter. The Hamming diameter of T is the rank of the complementary set $E(G) \setminus T$ of edges, while the classical diameter [**3**] is the maximal length of the unique path connecting some two vertices of T and using exclusively the edges of T.
4. Hamming distance among the bases of a matroid coincides with the classical distance among vertices of the associated matroid base graph, introduced and investigated in detail by S. B. Maurer ([**5**] and [**6**]). The vertices of a matroid base graph are adjoined to the matroid bases and edges to the neighbouring pairs of bases in the sense of the exchange operation.

4. TWO FAMILIES OF MATROIDS WITH ARBITRARILY WIDE RANGE OF BASE DIAMETERS

Two infinite families of matroids are described in this section. A matroid with any prescribed width of range of base diameters can be found in each one of these families.

The first family consists of graphic matroids induced by a family of connected graphs shown in Figure 1. Bases of these graphic matroids correspond to spanning trees and one can speak about "diameters of spanning trees".

The second family consists of matroids which, except for the members with the smallest ranks 0 and 2, are neither graphic, nor binary. The matroids of this family belong to a special class of matroids called E-chains.

4.1. Graphic case

Let $G_n,\ n = 0, 1, 2, ...$ denote the graph with $2n+2$ vertices shown in Figure 1, where n denotes the number of cycles of length 2:

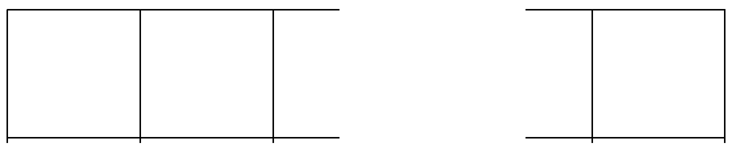

Figure 1. The graph G_n

Theorem 2. *The graph G_n has range $[n, 2n]$ of spanning tree diameters.*

Proof. A spanning tree T of G_n has diameter d if and only if exactly $d-n$ edges of T belong to cycles of length 2. Since obviously there exist spanning trees of G_n having from 0 to n edges belonging to cycles of length 2, it follows that the diameters of these spanning trees cover the whole interval $[n, 2n]$. □

EXAMPLE. Consider the graph G_2 (Figure 2). The graph G_2 has 35 bases, 1 of diameter 2, 10 of diameter 3 and 24 of diameter 4. The only base of diameter 2 is 34589. The bases of diameter 3 are obtained as unions of the edge-set 89 with some of the edge-sets 134, 135, 145, 234, 235, 245 and of the edge-set 345 with some of the edge-sets 68, 69, 78, 79. Finally, the bases of diameter 4 are obtained as unions of the edge-sets 134, 135, 145, 234, 235, 245 with some of the edge-sets 68, 69, 78, 79.

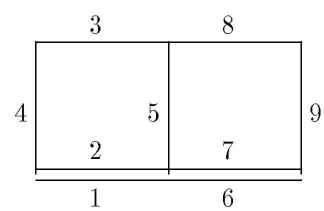

Figure 2. The graph G_2

4.2. Non-graphic case

A *cyclic flat* of a matroid on a ground-set E is a subset X of E which satisfies that $rank(X \setminus x) = rank(X)$ for each $x \in X$ and $rank(X \cup y) = rank(X) + 1$ for each $y \in E \setminus X$. It is well-known [**8**] that cyclic flats of a matroid, ordered by inclusion, constitute a general finite lattice. An *E-chain* [**1**]is a matroid that has the lattice of cyclic flats isomorphic to a chain. There are exactly 2^n non-isomorphic E-chains on an n-point ground-set.

Let $M_n,\ n=0,1,2,...$ denote an E-chain matroid of rank $2n$, defined on a ground-set X of cardinality $4n$ as follows: there are exactly $n+2$ cyclic flats of M_n, that constitute the chain $\emptyset = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset X$, while cardinalities and ranks of the cyclic flats $X_1, X_2, \ldots, X_n, X$ are equal to $2n,\ 2n{+}2, \ldots, 4n{-}2,\ 4n$ and $n,\ n+1,\ \ldots,\ 2n-1, 2n,$ respectively.

Matroid M_n is non-binary (hence also non-graphic) for $n \geq 2$, since it contains a minor isomorphic to the rank 2 uniform matroid on 4 elements [**9**]; it suffices to consider the interval $[X_0, X_1]$.

Cyclic flats $X_1, \ldots,\ X_{n-1},\ X_n,\ X$ define a partition of the ground-set X into $n+1$ disjoint subsets $X_1,\ X_2 - X_1, \ldots, X_n - X_{n-1},\ X - X_n$ of cardinalities $2n, 2, \ldots, 2, 2$ respectively. Let b denote a subset of X of cardinality $2n$ and let $(b_1, b_2, \ldots, b_n, b_{n+1})$ be the $(n+1)$-tuple of non-negative integers associated with b, so that b_i denotes the cardinality of the intersection of b with the i-th subset of the considered partition. It is readily seen that a subset b is a base of M if and only if the following set $(*)$ of conditions is satisfied:

$$(*)\qquad \begin{gathered} b_1 \leq n,\ b_2 \leq 2, \ldots,\ b_n \leq 2,\ b_{n+1} \leq 2, \\ b_1 + b_2 \leq n+1, \ldots, b_1 + \cdots + b_n \leq 2n-1,\ b_1 + \cdots + b_n + b_{n+1} \leq 2n. \end{gathered}$$

Theorem 3. *The matroid M_n has range $[n, 2n]$ of base diameters.*

Proof. Let d be a natural number satisfying $n \leq d \leq 2n$. It is easy to check that the $(n+1)$-tuple $(b_1, b_2, \ldots, b_n, b_{n+1}) = (d-n, \underbrace{2, \ldots, 2}_{2n-d}, \underbrace{1, \ldots, 1}_{d-n})$ satisfies the conditions $(*)$ required for the bases. There exist exactly $\binom{2n}{b_1} \cdot \binom{2}{b_2} \cdots \binom{2}{b_n} \cdot \binom{2}{b_{n+1}}$ bases of M_n with this associated $(n+1)$-tuple; let b be any of them. We shall prove that the diameter of b (= rank of b^*) is equal to d. The $(n+1)$-tuple associated with b^* and w.r.t. the considered partition is equal to $(3n-d, \underbrace{0, \ldots, 0}_{2n-d}, \underbrace{1, \ldots, 1}_{d-n})$.

Although $3n - d \geq n$, the rank of a subset of cardinality $3n-d$ of X_1 is equal to n. Addition of $d-n$ elements from some further $d-n$ subsets of the partition necessarily augments the rank for exactly $d-n$, thus making the rank of b^* equal to d. $\Box$

Example. The cyclic flats of the non-binary matroid M_2 on the ground-set 12345678 are $\emptyset$, 1234, 123456 and 12345678 of ranks 0, 2, 3, 4, respectively.

The matroid M_2 has 47 bases, 1 of diameter 2, 22 of diameter 3 and 24 of diameter 4. The only base of diameter 2 is 5678. Eight bases of diameter 3 correspond to the triple $(b_1, b_2, b_3) = (1, 1, 2)$, another eight to the triple $(1, 2, 1)$ and six to the triple $(2, 0, 2)$. The 24 bases of diameter 4 have two elements from the set 1234, one element from the set 56 and one element from the set 78; that is, all these bases correspond to the triple $(b_1, b_2, b_3) = (2, 1, 1)$.

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