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ON THE LOGARITHMIC CONCAVITY OF $(r-1)\zeta(r)$

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Inequality (1) is proved.

In this note we show that the sequence $\{(r-1)\zeta(r)\},\ r=1,2,\dots$, where $\zeta(r)$ is the zeta function $\sum\limits_{k=1}^{+\infty} 1/k^r$, is logarithmically concave, i.e.,

$$r^2/(r^2-1) > \zeta(r)\zeta(r+2)/(\zeta(r+1))^2. \tag{1}$$

This result arose in generalizing an inequality of the first author which appeared recently as a proposed problem [**1**].

For the proof we use upper and lower bounds for $\zeta(r)$ as abtained by use of the Maclaurin integral test applied to the function $h(x)=1/x^r$. Since $h(x)$ is strictly decreasing on $[1,+\infty)$, the following inequalities hold:

$$\zeta(r+2) < \frac{1}{1^{r+2}} + \frac{1}{2^{r+2}} + \int\limits_2^{+\infty} \frac{\mathrm{d}x}{x^{r+2}} = 1 + \frac{1}{2^{r+2}} + \frac{1}{(r+1)2^{r+1}} < 1 + \frac{1}{2^{r+1}}\,,$$

$$\zeta(r+1) > \frac{1}{1^{r+1}} + \int\limits_2^{+\infty} \frac{\mathrm{d}x}{x^{r+1}} = 1 + \frac{1}{r\cdot 2^r}\,.$$

Using the latter bounds, we first show that (1) is valid for $r\ge 7$. It sufficies to establish the stronger inequality

$$\frac{r^2}{(r^2-1)} > \left(1+\frac{1}{2^{r+1}}\right)\left(1+\frac{1}{2^{r-1}}\right)\Big/\left(1+\frac{1}{r\cdot 2^{r-1}}+\frac{1}{r^2\cdot 2^{2r}}\right)$$

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or equivalently that

$$2^r - 2(r^2 - r - 1) - \frac{r^2 - 1}{2} - \frac{r^2 - 2}{2^r} > 0$$

and which holds for $r \geq 7$. The causes for $r = 2, 3, 4, 5, 6$ folow by numerical evaluation using the known values of $\zeta(r)$ for these values.

It will be shown in a subsequent note that function $(r-1)\zeta(r)$ is logarithmically concave for all $r \geq 2$ and that the function

$$((r)\zeta(r+1))^2 - ((r-1)\zeta(r))((r+1)\zeta(r+2))$$

is increasing in r and has limit 1 as $r \to +\infty$.

REFERENCES

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