TWO INEQUALITIES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Inequalities (2) and (4) are proved.

An inequality that appear in Mitrinović [1, p. 247] is

\[ \frac{3x}{2 + \sqrt{1 - x^2}} \leq \sin^{-1} x, \quad 0 \leq x \leq 1. \]  (1)

This is best possible at \( x = 0 \) but not at 1. It occurred to me to see if there is a corresponding upper bound for \( \sin^{-1} x \). Indeed there is,

\[ \sin^{-1} x \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}, \quad 0 \leq x \leq 1 \]  (2)

and equality holds at both ends of the interval. We give a proof of both inequalities. Let

\[ f_\alpha(x) = \frac{ax}{2 + \sqrt{1 - x^2}} = \sin^{-1} x, \quad 3 \leq a \leq \pi, \quad 0 \leq x \leq 1. \]

Note that \( f_\alpha(0) = 0 \) and \( f_\alpha(1) = \frac{a - \pi}{2} \leq 0 \) with \( f_\alpha(1) = 0 \). Then

\[ f'_\alpha(x) = \left( \sqrt{1 - x^2} \left( 2 + \sqrt{1 - x^2} \right)^2 \right)^{-1} \left( (2a - 4) \sqrt{1 - x^2} + a - 5 + x^2 \right). \]  (3)

Call the second bracket \( g_\alpha(x) \equiv (2a - 4) \sqrt{1 - x^2} + a - 5 + x^2 \) then \( g'_\alpha(a) = \frac{2x}{\sqrt{1 - x^2}} \left( \sqrt{1 - x^2} - (a - 2) \right) \neq 0 \) on \((0,1)\). So \( g \) is monotone decreasing. For \( a = 3 \), \( g_3(0) = 0 \) so \( g_3(x) \leq 0 \) and \( f'_3(2) \leq 0 \) as required to prove (1). For \( a = \pi \), \( g_\pi(x) \) has one zero and so \( f_\pi(x) \) is unimodal with \( f'_\pi(0) > 0 \) and \( f_\pi(0) = f_\pi(1) = 0 \). This proves (2).
Our second inequality concerns log convex functions. If \( f > 0 \) and \( \log f \) is convex on \( \mathbb{R} \) then

\[
\frac{\pi}{4} \int_{-1}^{1} f(x + vt) \cos \frac{\pi t}{2} \, dt \leq \frac{f(x + v) + f(x - v)}{2}, \quad x, v \in \mathbb{R}.
\]

The proof is to write

\[
f(x + tv) \leq f(x + v)^{\frac{1 + t}{2}} f(x - v)^{\frac{1 - t}{2}}, \quad -1 \leq t \leq 1.
\]

Let \( B = \frac{1}{2} \log \frac{f(x + v)}{f(x - v)} \), then

\[
\frac{\pi}{4} \int_{-1}^{1} f(x + vt) \cos \frac{\pi t}{2} \, dt \leq (f(x + v) f(x - v))^{1/2} \frac{\pi}{4} \int_{-1}^{1} \cos \frac{\pi t}{2} e^{Bt} \, dt
\]

\[
= \frac{\pi^{2}}{8} \frac{e^{B} + e^{-B}}{B^{2} + \frac{\pi^{2}}{4}} (f(x + v) f(x - v))^{1/2} = \frac{\pi^{2}}{B^{2} + \frac{\pi^{2}}{4}} f(x + v) f(x - v) \frac{1}{2}.
\]

Equality holds in this argument if \( f(x) \) is linear. In any case, (4) follows. Equality holds in (4) if \( f \) is a constant.

Moreover, if \( f \) is log concave we have

\[
\frac{\pi}{4} \int_{-1}^{1} f(x + vt) \cos \frac{\pi t}{2} \, dt \geq \frac{f(x + v) + f(x - v)}{2} \left( \frac{\pi^{2}}{4} + \frac{\pi^{2}}{4} \log \frac{f(x + v)}{f(x - v)} \right).
\]

REFERENCES


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