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ANOTHER EXTENSION OF THE MITRINOVIĆ–ĐOKOVIĆ INEQUALITY

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

A certain extension of the Mitrinović–Đoković Inequality [2, p.282] is proved by an elementary method.

1. INTRODUCTION

In [**2**, p. 282–283], D. S. MITRINOVIĆ and D. Ž. ĐOKOVIĆ extended an inequality given in [**1**, p. 39]:

$$\left(x_1+\frac{1}{x_1}\right)^2+\left(x_2+\frac{1}{x_2}\right)^2\geq\frac{25}{2},\qquad x_1,\,x_2>0,\qquad x_1+x_2=1$$

to the following

$$\sum_{k=1}^{n}\left(x_k+\frac{1}{x_k}\right)^a\geq\frac{(n^2+1)^a}{n^{a-1}},\tag{1}$$

where $a,\,x_1,\ldots,x_n>0,\ x_1+\cdots+x_n=1$.

J. CHEN [**3**] extended the inequality (1) in the following way. Let $x_k>0,\ x_1+\cdots+x_n=s\leq n-2+2\sqrt{2+\sqrt{5}}$, and $a>0$, then

$$\sum_{k=1}^{n}\left(x_k+\frac{1}{x_k}\right)^a\geq n\left(\frac{s}{n}+\frac{n}{s}\right)^a.\tag{2}$$

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A further generalization is obtained by Z. Wang and J. Chen [**4**].

Let $a \geq -1$, $x_1, \dots, x_n > 0$, $x_1 + \dots + x_n = s$, and when $a < 1$, $s \leq n - 2 + 2\left(\dfrac{2 - a + \sqrt{5 - 4a}}{1 - a}\right)^{1/2}$. Then

$$M_a\left(x + \frac{1}{x}\right) \geq \frac{s}{n} + \frac{n}{s}, \tag{3}$$

where

$$M_a\left(x + \frac{1}{x}\right) = \begin{cases} \left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\left(x_k + \dfrac{1}{x_k}\right)^a\right)^{1/a}, & a \neq 0; \\ \left(\displaystyle\prod_{k=1}^{n}\left(x + \dfrac{1}{x}\right)\right)^{1/n}, & a = 0. \end{cases}$$

Equality is valid if and only if $x_1 = \dots = x_n$.

The object of this paper is to show that (see [**5**, p. 37])

Theorem. *Let* $x_k > 0$, $x_1 + \dots + x_n = s < \sqrt{n^2 - 1} + \sqrt{2n - 1}$, *and* $a \geq -1$, *then the inequality* (3) *is valid, with equality if and only if* $x_1 = \dots = x_n$.

2. LEMMA

Lemma. *If positive numbers x and y satisfy* $(n - 1)x + y = s < \sqrt{n^2 - 1} + \sqrt{2n - 1}$, $n \geq 2$, *then*

$$(n - 1)\left(x + \frac{1}{x}\right)^{-1} + \left(y + \frac{1}{y}\right)^{-1} < n\left(\frac{s}{n} + \frac{n}{s}\right)^{-1}, \tag{4}$$

and $\sqrt{n^2 - 1} + \sqrt{2n - 1}$ *is the greatest value s such that* (4) *is valid. The equality in* (4) *holds if and only if* $x = y$.

Proof. Using the following identity

$$\begin{aligned} &n\left(\frac{s}{n} + \frac{n}{s}\right)^{-1} - (n - 1)\left(x + \frac{1}{x}\right)^{-1} - \left(y + \frac{1}{y}\right)^{-1} \quad (5) \\ &= \frac{n^2((n-1)x + y)}{((n-1)x + y)^2 + n^2} - \frac{(n-1)x}{x^2 + 1} - \frac{y}{y^2 + 1} \\ &= \frac{(n-1)(x-y)^2\big(((2n-1)x + (n+1)y) - xy((n-1)x + y)\big)}{\big(((n-1)x + y)^2 + n^2\big)(x^2 + 1)(y^2 + 1)} \\ &= \frac{(n-1)(x-y)^2\big(s((n-1)(x-y)^2 + 2n^2) + (n^2 - s^2)(x + (n-1)y)\big)}{n^2(s^2 + n^2)(x^2 + 1)(y^2 + 1)} \end{aligned}$$

$$= \Big((x-y)^2\big(((n-1)^2x^2-y^2-n(n-2))^2$$
$$-(s^2-(\sqrt{n^2-1}+\sqrt{2n-1})^2)(s^2-(\sqrt{n^2-1}-\sqrt{2n-1})^2)\big)\Big):$$
$$:\Big(4s(s^2+n^2)(x^2+1)(y^2+1)\Big)$$

and $n > \sqrt{n^2-1}-\sqrt{2n-1}$, we obtain (4) for $s \leq \sqrt{n^2-1}+\sqrt{2n-1}$.

For any $\delta > 0$, putting $s^2 = \left(\sqrt{n^2-1}+\sqrt{2n-1}\right)^2+\delta$ in the last $(\,\cdot\,)$, it becomes

$$((n-1)^2x^2-y^2-n(n-2))^2-\delta\big(\delta+4\sqrt{(n^2-1)(2n-1)}\,\big). \tag{6}$$

When $x = (s^2+n(n-2))/(2(n-1)s)$, $y = (s^2-n(n-2))/(2s)$, it is negative. Hence $\sqrt{n^2-1}+\sqrt{2n-1}$ is the best possible.

3. PROOF OF THE THEOREM

Using the monotonicity of the power means, we only necessarily prove

$$M_{-1}\left(x+\frac{1}{x}\right) \geq \frac{s}{n}+\frac{n}{s}, \tag{7}$$

i.e.

$$\left(x_1+\frac{1}{x_1}\right)^{-1}+\cdots+\left(x_n+\frac{1}{x_n}\right)^{-1} \leq n\left(\frac{s}{n}+\frac{n}{s}\right)^{-1}. \tag{8}$$

We will use induction to show (8). At first, from the lemma, (8) holds in the case $n = 2$. Now assume that (8) holds for some $n-1$ $(n \geq 2)$.

If $x_1 \leq \cdots \leq x_n$, then $x_1+\cdots+x_{n-1} \leq \dfrac{n-1}{n}\left(\sqrt{n^2-1}+\sqrt{2n-1}\right) \leq \sqrt{n^2-2n}+\sqrt{2n-3}$. By the assumption and the lemma, we have

$$\begin{aligned}
&\left(x_1+\frac{1}{x_1}\right)^{-1}+\cdots+\left(x_n+\frac{1}{x_n}\right)^{-1} \\
&\leq (n-1)\left(\frac{x_1+\cdots+x_{n-1}}{n-1}+\frac{n-1}{x_1+\cdots+x_{n-1}}\right)^{-1}+\left(x_n+\frac{1}{x_n}\right)^{-1} \\
&\leq n\left(\frac{x_1+\cdots+x_n}{n}+\frac{n}{x_1+\cdots+x_n}\right)^{-1} = n\left(\frac{s}{n}+\frac{n}{s}\right)^{-1}.
\end{aligned} \tag{9}$$

Therefore, we get that (8) is true for arbitrary n.

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