Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 6 (1995), 12–25.

TRANSFORMATION AND FACTORIZATION OF SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS AND ITS IMPLEMENTATION IN REDUCE

Lev M. Berkovich, Fedor L. Berkovich

Dedicated to the memory of Professor Dragoslav S. Mitrinović

In the present paper the authors propose an algorithmic procedure different from the known algorithms [1,2] of searching for the Liouvillian solutions of linear ordinary differential equations (LODE) of the second order. This procedure is based upon the reducibility of LODE of the second order to the equation with constant coefficients by Kummer-Liouville transformation [3]. It is applicable not only to the equations with rational coefficients, but also to the equations with arbitrary functional coefficients and parameters.
Short description of the package structure is given in [4,5]. However in this paper we do not consider, in general, the equations having algebraic and special functions as the solutions, as well as the equations having solutions built by Euler-Darboux-Imshenetsky transformation.

The package testing was carried out using treats [6,7,8]

1. INTRODUCTION

The construction of algorithms for finding formal solutions for some classes of equations is the main purpose of any constructive theory of ordinary differential equations theory. Explicit formulas are very important and include in themselves all the available information. It is also necessary to have them to develop our mathematical and scientifical intuition and to compare different theories including the bounds of their applicability.

Euler, Liouville, Kummer, Jacobi and other mathematicians discovered that the basic method of integration and investigation of differential equations implies convenient changes of variables which reduce the original equations to a simpler form. However, they didn't offer the algorithms for finding this trans-

[0]1991 Mathematics Subject Classification: Primary 34A05, Secondary 68-04

formations and, as a result, the search for suitable substitutions had an heuristic character.

On the other hand, closed relations between linear ODE and algebraic equations have been known for a long time and they have resulted in method of factorization of differential operators. Here many results had a non-constructive character, too.

One of the authors developed in [**3**] the effective method of transformation of differential equations in which both the change of variables and the factorization of differential operators were used.

There exist different approaches to receiving the Liouvillian solutions of ODE, i.e. those that may be represented in a finite form using elementary, algebraic functions and quadratures.

An interesting though not complete, review of various methods of obtaining the Liouvillian and other formal solutions of ODE may be found in [**9**].

Modern computer algebra systems are becoming the powerful means of PC implementation of the exact methods of ODE investigation and integration. As a result, many users are able now to use in practice those methods that formerly were available only for specialists.

In this paper we propose our algorithm for finding the general solutions of nonhomogeneous second order LODE

$$Ly \equiv a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = f(x),$$

where coefficients a_2, a_1, a_0 belong to some differential field K and they are arbitrary differentiable functions, possibly, containing parameters. The heart of an algorithm is the search for variable change that reduces the corresponding homogeneous equation $Ly = 0$ to one with constant coefficients.

Algorithm is implemented in computer algebra system REDUCE [**10**].

2. METHOD

2.1. SEARCH FOR THE KUMMER-LIOUVILLE TRANSFORMATION

Let the equation

$$y'' + a_1(x)\,y' + a_0(x)\,y = 0, \quad a_1 \in \mathbf{C}^1(I),\ a_0 \in \mathbf{C}(I), \tag{1}$$

where $I = \{x | a < x < b\}$ be given. Here for simplicity we assumed $a_2 = 1$. Let us apply here the *Kummer-Liouville* (KL) *transformation*, i.e. the variable change:

$$y = v(x) z(t),\ \mathrm{d}t = u(x)\mathrm{d}x, \quad u, v \in \mathbf{C}^2(I),\ uv \neq 0,\ \forall x \in I. \tag{2}$$

Due to the Stäckel-Lie theorem (2) is the most general transformation that keeps the order and linearity of an equation (1). It is the basic part of the Kummer problem (see [**11**]) of reduction of (1) to the equation of a form

$$z''(t) + b_1(t)\,z'(t) + b_0(t)\,z(t) = 0, \quad b_1 \in \mathbf{C}^1(J),\ b_0 \in \mathbf{C}(J). \tag{3}$$

where $J = \{t | c < t < d\}$.

The Kummer problem is always solvable [**12**] and, therefore, there always exists the KL-transformation that reduces (1) to (3). However, the problem of reduction of (1) to the equation with constant coefficients

$$z''(t) + b_1\,z(t) + b_0\,z(t) = 0, \quad b_1, b_0 = \mathrm{const}, \tag{4}$$

is of the most essential interest.

The KL-transformation may be found using

Lemma 1. *The equation* (1) *may be reduced to* (4) *by the* KL*-transformation, for which the kernel* $u(x)$ *satisfies the second order Kummer-Schwarz equation*

$$\frac{1}{2}\,\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{4}\,\delta u^2 = A_0(x), \tag{5}$$

where

$$A_0(x) = a_0 - \frac{1}{4}\,{a_1}^2 - \frac{1}{2}\,a_1'$$

is a semi-invariant of (1), $\delta = {b_1}^2 - 4b_0$ *is a discriminant of a characteristic equation*

$$r^2 + b_1 r + b_0 = 0. \tag{6}$$

Then, the multiplier $v(x)$ *of* KL*-transformation satisfies the explicit equation*

$$v(x) = |u|^{-1/2} \exp\Big(-\frac{1}{2}\int a_1\,\mathrm{d}x\Big)\exp\Big(\frac{1}{2}\,b_1\int u\,\mathrm{d}x\Big)$$

and, moreover, $v(x)$ *and* $u(x)$ *are related by a differential equation*

$$v'' + a_1(x)\,v' + a_0(x)\,v - b_0 u^2 v = 0. \tag{7}$$

Lemma 1 allows to find a constructive approach to finding $u(x)$ and, consequently, the Kummer–Liouville transformation (for more details see section 3).

Let us consider the equation

$$y_1'' + a_1\,y_1' + a_{01}\,y_1 = 0, \quad a_{01} = a_0(x) - b_0\,u(x)^2 \tag{8}$$

resulted from (7) after the change $v \to y_1$. This equation and (1) have the same kernel $u(x)$ of the KL-transformation. Since the coefficient $a_0(x)$ due to the formula

$$a_0(x) = a_{01}(x) + b_0\,u(x)^2$$

contains $u(x)^2$ as an additive part (up to the multiplicative constant $b_0 \neq 0$), then it is reasonable to choose variants of $u(x)$ among the expressions $a_0(x)$ or $A_0(x)$ or their additive parts.

Then the function $w(x) = u(x)^2$ must satisfy the equation

$$\frac{1}{4}\frac{w''}{w} - \frac{5}{16}\left(\frac{w'}{w}\right)^2 - \frac{1}{4}\delta w = A_0.$$

REMARK: In order to apply these statements, at least one of the expressions δ or b_0 should not be zero.

2.2. FACTORIZATION

Let us say that the equation (1) *admits factorization* if its differential operator

$$L = D^2 + a_1 D + a_0, \quad D = \mathrm{d}/\mathrm{d}x$$

may be represented as a product of first order operators

$$L = (D - \alpha_2)(D - \alpha_1), \quad \alpha_1 = \alpha_1(x),\ \alpha_2 = \alpha_2(x) \tag{9}$$

Here the differential analog of VIÈTE formulas

$$a_1 = -(\alpha_1 + \alpha_2),\ a_0 = \alpha_1\alpha_2 - a_1' \tag{10}$$

remains valid.

Due to the G.MAMMANA's theorem [**13**], (1) always admits factorization.

Lemma 2. *The equation* (1) *may be reduced to* (4) *by the transformation* (2) *and admits factorization*

$$L = \left(D - \frac{v'}{v} - \frac{u'}{u} - r_2 u\right)\left(D - \frac{v'}{v} - r_1 u\right) y = 0,$$

where r_1, r_2 are roots of the characteristic equation (7).

Lemma 3. *Zeros* (*roots*) *of factorization may be represented in the form*

$$\alpha_1 = -\frac{1}{2}\frac{u'}{u} - \frac{1}{2}a_1 + \frac{\sqrt{\delta}}{2}u, \quad \alpha_2 = \frac{1}{2}\frac{u'}{u} - \frac{1}{2}a_1 - \frac{\sqrt{\delta}}{2}u.$$

This formula connects "zeros" of factorization with KUMMER-LIOUVILLE transformation.

Let us now find the feedback of KL-transformation with "zeros" of factorization. Using the arbitrariness of determining the characteristic roots r_1, r_2, we may require $r_1 = r_2 = 0$. Then the above formula takes a form

$$\alpha_1 = \frac{v'}{v}, \quad \alpha_2 = \frac{v'}{v} + \frac{u'}{u}$$

from where

$$v = e^{\int a_1 \mathrm{d}x}, \quad u = e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}. \tag{11}$$

So, we may formulate

Lemma 4. *The equation* (1) *by transformation*

$$y = e^{\int \alpha_1 \,\mathrm{d}x}, \quad \mathrm{d}t = e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}\,\mathrm{d}x$$

may be reduced to the equation with constant coefficients

$$z''(t) = 0.$$

Since in this case the KL-transformation is represented using the "roots" of factorization, an elementary procedure for finding factorization may be specified.

Let us consider the Kummer-Schwarz equation (5) that due to (10) will take a form

$$A_0 = -\frac{1}{4}\,(\alpha_2 - \alpha_1)^2 - \frac{1}{2}\,(\alpha_2 - \alpha_1)'$$

from where

$$\alpha = \alpha_2 - \alpha_1.$$

So, we may find α as $\alpha = -2\sqrt{w}$, where w is an a additive part of semi-invariant A_0.

2.3. FUNDAMENTAL SYSTEM OF SOLUTIONS OF LODE

Case 1: *Equation* (1) *admits the factorization* (11).

Lemma 5. *Equation* (1) *has the following fundamental system of solutions* (FSS):

$$\begin{aligned} y_{1,2} &= u^{-1/2} \exp\left(-\frac{1}{2}\int a_1 \,\mathrm{d}x\right) \exp\left(\pm\frac{\sqrt{\delta}}{2}\int u \,\mathrm{d}x\right), \quad \delta \neq 0, \\ y_1 &= u^{-1/2} \exp\left(-\frac{1}{2}\int a_1 \,\mathrm{d}x\right), \quad y_2 = y_1 \int u \,\mathrm{d}x, \quad \delta = 0. \end{aligned}$$

Case 2: $b_1 = b_0 = 0$.

Lemma 6. *If the factorization* (14) *is known, then* FSS *is*:

$$y_1 = e^{\int \alpha_1 \,\mathrm{d}x}, \quad y_2 = y_1 \int e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}\,\mathrm{d}x.$$

2.4. PARTIAL SOLUTION

Let the nonhomogeneous equation

$$y'' + a_1\, y' + a_0\, y = f(x)$$

be given. If the FSS $\{y_1, y_2\}$ of correspondent equation (1) is known, then the partial solution y^* for Lemma 6 has the form

$$y^* = -y_1 \int e^{\int a_1\,\mathrm{d}x}\, y_2 f\,\mathrm{d}x + y_2 \int e^{\int a_1\,\mathrm{d}x}\, y_1 f\,\mathrm{d}x$$

while for Lemma 5

$$y^* = \frac{1}{2\sqrt{-b_0}} \left(y_1 \int e^{\int a_1\,\mathrm{d}x}\, y_2 f\,\mathrm{d}x - y_2 \int e^{\int a_1\,\mathrm{d}x}\, y_1 f\,\mathrm{d}x \right), \quad b_0 \neq 0.$$

2.5. SEMI-INVARIANTS AND SPECIAL CASES OF KUMMER–LIOUVILE TRANSFORMATION

2.5.1. Semi-invariant J_0 (invariant by the transformation of dependent variable).

As mentioned above, J_0 has the form

$$J_0 = a_0 - \frac{1}{4}{a_1}^2 - \frac{1}{2}a_1'.$$

If J_0 =const, then the KL-transformation will take the form

$$y = \exp\left(-\frac{1}{2}\int a_1\,\mathrm{d}x\right) z, \quad \mathrm{d}t = \mathrm{d}x.$$

Factorization of operator L in this case will become commutative:

$$L = \left(D + \frac{1}{2}\,a_1 + \sqrt{b_0}\right)\left(D + \frac{1}{2}\,a_1 - \sqrt{b_0}\right).$$

2.5.2. Semi-invariant J_1 (invariant by the transformation of independent variable) [14].

$$J_1 = a_0 e^{2\int a_1\,dx} \left(b_1 \int e^{-\int a_1\,\mathrm{d}x}\,\mathrm{d}x + c \right)^2. \tag{12}$$

If J_1 =const, then the KL-transformation will take the form

$$y = z, \quad \mathrm{d}t = -\frac{e^{-\int a_1\,\mathrm{d}x}}{b_1 \int e^{-\int a_1\,\mathrm{d}x}\,\mathrm{d}x + c}\,\mathrm{d}x.$$

We may determine whether J_1 is a constant or not, using the formula

$$\frac{a_1}{\sqrt{|a_0|}} + \frac{1}{2}\frac{a_0'}{a_0\sqrt{|a_0|}} = b_1 = \text{const}.$$

2.6. EQUATIONS SOLVABLE ALGEBRAICALLY

2.6.1. Exponential solutions

Let the equation $Ly = 0$ have an exponential solution $y = e^{\lambda x}$, where λ =const. Then the characteristic equation

$$r^2 + a_1(x)\,r + a_0(x) = 0$$

has among its roots r_1, r_2 not a function but a number λ:

$$r_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{{a_1}^2}{4} - a_0}$$

Factorization L takes the form:

$$L = (D + a_1 + \lambda)(D - \lambda).$$

2.6.2. Adjoint equations

By definition, the *adjoint* for

$$Ly = a_2 y'' + a_1 y' + a_0 y = a_2(D - \alpha_2)(D - \alpha_1)y = 0$$

equation is an equation

$$L^* y \equiv a_2\,y'' + (2a_2' - a_1)\,y' + (a_0 - a_1' + a_2'')\,y = 0$$

it admits factorization

$$L^* = (D + \alpha_1)(a_2 D + a_2' + a_2\alpha_2) = a_2\left(D + \frac{a_2'}{a_2} + \alpha_1\right)\left(D + \frac{a_2'}{a_2} + \alpha_2\right).$$

Let us consider the characteristic equation for an adjoint one:

$$a_2\, r^2 + (2a_2' - a_1)r + a_0 - a_1' + a_2'' = 0.$$

If one of its roots is a number λ, then the factorization L^* takes the form:

$$L^* = a_2 \left(D + \frac{a_2'}{a_2} + \alpha_1 \right) (D - \lambda).$$

Simultaneously, the factorization L is of the form

$$L = a_2 \left(D + \frac{a_2'}{a_2} + \lambda \right) \left(D - \frac{a_2'}{a_2} - \alpha_1 \right).$$

2.6.3. Exact equation

If $\lambda = 0$, then we have *an exact equation*, for which the factorization L has a form

$$L = D(a_2 D - \alpha_1 a_2) = a_2 \left(D + \frac{a_2'}{a_2} \right) (D - \alpha_1).$$

3. FOUNDATIONS OF AN ALGORITHM

3.1. RELATED SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

The search algorithm for the Liouvillian solutions is concerned with a procedure of "generation" of integrable equations [**15**]. We shall call *related equations* the following two second order equations: equation (1) and equation (8), which are connected by a Kummer-Liouville transformation (see below).

Lemma 5. *Kummer-Schwarz equation* (5) *has the following general solution depending on* δ :

$$\begin{aligned}
u_1(x) &= F(\alpha_1 Y_2 + \beta_2 Y_1)^{-1}(\alpha_2 Y_2 + \beta_2 Y_1)^{-1}, \quad \delta = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 < 0 \\
u_2(x) &= F(A{Y_2}^2 + BY_1Y_2 + C{Y_1}^2)^{-1}, \quad \delta = B^2 - 4AC > 0 \\
u_3(x) &= F(\alpha Y_2 + \beta Y_1)^{-2}, \quad \delta = 0
\end{aligned}$$

Let us also point out here some special cases:

$$\begin{aligned}
u_4(x) &= F(\alpha Y_2 + \beta Y_1)^{-1} Y_1^{-1}, \text{ or } = F(\alpha Y_2 + \beta Y_1)^{-1} Y_2^{-1}, \quad \delta = \alpha^2; \\
u_5(x) &= F Y_1^{-2}, \text{ or } = F Y_2^{-2}, \text{ or } = F(Y_1 Y_2)^{-1}, \quad \delta = 0
\end{aligned}$$

Here

$$F = e^{-\int a_1\, \mathrm{d}x}$$

and Y_1, Y_2 form the fundamental system of solutions of an equation (1).

Below for the description of a set of related integrable equations let us consider the equation of a form:

$$y'' + a(x)\, y = 0.$$

Theorem. *This equation generates the sequence*

$$\begin{aligned}
&y''_{k+1} + a_{k+1}\, y_{k+1} = 0,\\
&a_{k+1} = a - \sum_{s=0}^{k+1} b_{0(s)} {u_s}^2,\\
&b_{0(s)} = \text{const}, \quad a_{k+1} = a_k - b_{0(k+1)} u_{k+1}^2, \quad b_{0(k+1)} \neq 0,
\end{aligned}$$

where $u(x)$ satisfies the following consequence of equations (KSH-2):

$$\begin{aligned}
&\frac{1}{2}\frac{u''_{(s)}}{u_{(s)}} - \frac{3}{4}\left(\frac{u'_{(s)}}{u_{(s)}}\right)^2 - \frac{1}{4}\,\delta_{(s)} u_{(s)}^2 = A_{(s-1)},\\
&\delta_{(s)} = b_{1(s)}^2 - 4 b_{0(s)}
\end{aligned}$$

is a determinant of a characteristic equation

$$r_s^{\,2} \pm b_{1(s)}\, r_s + b_{0(s)} = 0.$$

Then the linear independent solutions $y_{(1,2)k+1}$ have a form:

$$\begin{aligned}
y_{(1,2)k+1} &= |u(k+1)|^{-1/2} \exp\left(\pm(1/2) b_{1(k+1)} \int u_{(k+1)}\, \mathrm{d}x\right), \; b_{1(k+1)} \neq 0,\\
y_{1(k+1)} &= |u(k+1)|^{-1/2}, \; y_{2(k+1)} = |u_{(k+1)}|^{-1/2} \int u_{(k+1)}\, \mathrm{d}x, \; b_{1(k+1)} = 0.
\end{aligned}$$

4. IMPLEMENTATION

The REDUCE program SOLDE which implements the algorithm described above, has the following main characteristics:

```
INPUT: a2,a1,a0,f          %coefficients of given equation
OUTPUT: u,v,               %variable change
        b1,b0,             %coefficients of reduced equation
        alpha1,alpha2,     %coefficients of factorization
        y1,y2,             %FSS
        yp                 %partial solution for nonhomogeneous ODE
```

The computer program SOLDE written in the system of computer algebra system REDUCE, involves a detailed consideration of some important procedures which will be mentioned in this section. From the factorization of differential operator we deduce that in case $a_2(x) \neq$const the square root of $a_2(x)$ and one of the factors of $a_2(x)$ may be tested as possible variants of $1/u(x)$.

Program SOLDE was tested on hundreds of equations [**6**,**7**,**8**] and it proved successful in 75% . Failure in the rest 25% is caused by three reasons: 1) Though KUMMER-LIOUVILLE transformation has already been included into the program, another important transformation, of EULER-DARBOUX-IMSHENETSKY, has not been included into the program yet; 2) The algorithm can fail to work when $b_1 = b_0 = 0$, 3) The given version of the program does not apply in general to the equations having algebraic and special functions as their solutions.

To illustrate the current possibilities of this program, let us name the sorts of equations from the demonstration file: Equations with constant coefficients, EULER's equations, equations with exponential, algebraic coefficients, trigonometrical, hyperbolic functions, with arbitrary functions, mixed and rational coefficients.

```
*** I.Equations with constant coefficients ***
        Y''+ A*Y'+ B*Y = F(X)
        Y''+ A*Y'+ A*A/4*Y = F(X)
*** II.Euler's equations ***
        Y'' + A/X * Y' + B/(X*X) * Y = F(X)/(X*X)
*** III.Equations of a form :   ***
        Y'' + A1(x) * Y' = F(X)
*** IY.Equations with exponential coefficients ***
        Y''+ A*Y'+ B*E**(2*A*X) * Y = F(X)
*** Y.Equations with trigonometrical functions ***
        Y''+ 2*A*COT(A*X) * Y'+ (B*B-A*A) * Y = 0
        Y''+ (M*M + A/SIN(2*M*X)**2) * Y = 0
*** YI.Equations with hyperbolic functions ***
        Y''+ 2*TANH(X) * Y' + B*Y = 0
        Y''+ (-M*M + A/(SINH(M*X)**4)) * Y = 0
*** YII.Equations with algebraic coefficients ***
        Y'' + 8*A/(5*(A*X+B))*Y'+C*(A*X+B)**
           (1/5)/(5*(A*X+B))*Y = 0
*** YIII.Equations with mixed coefficients ***
        Y'' + 2*A/X*Y' +((B*B*E**(2*C*X)-1/4)*C*C+
           A*(A-1)/(X*X)) * Y = 0
        Y'' + (1/(4*X*X) + 1/(X*X)/(P*LOG(X)+Q)**4))
           * Y = 0
*** IX.Equations with arbitrary functions ***
        Y'' + 2*F(X)*Y' + (F(X)*F(X)+F'(X)+G''(X)/2/G(X)-
          3/4*G'(X)*G'(X)/G(X)*G(X)-A*G(X)*G(X))*Y=0
*** X.Equations with rational coefficients ***
        Y'' + D/(A*X*X+B*X+C)**2 * Y = 0
        Y''+(-M*(M+1)/(X*X) +(1/(-P/(2*M+1)*X**(-M)+
         Q*X**(M+1))**4)*Y=0
```

High efficiency of this implementation is caused by the fact that entire package was developed by enlarging the procedure SOLDE which allows the user to investigate his ODE from different points of view, and even if the program itself doesn't give an answer, it often helps the user to find the approaches to the solution of his equation making his operations more effective and faster. It also helps him test his propositions. Thus,each ODE may be investigated during the whole session by the following procedures:

SOLDE(a2,a1,a0,f)	solve the equation and display SUMMARY; automatically put the new equation
VERFAC(alpha1) VERSOL(y1) VERTRANS(u,v)	verify the given factorization, and if it is correct, display SUMMARY; similar to VERFAC - for solution - " - - for transformation
PUTEQ(a2,a1,a0,f,J0)	put the new equation without its solving by the procedure SOLDE; normally you would enter J0=0 and then semi-invariant would be computed by program; if you however aren't satisfied with its value, you enter the value yourself.
NORM(W)	normalize W, i.e. delete constant factor
SINV()	display the meanings of semi-invariants
SUMMARY()	display all the available information on the current equation,including the equation itself
HELP() INFO()	display brief and complete information on the package

Example of this program in action is given below:

```
**************************************************************

*       SOLVER OF LODE a2(x)*Y''+ a1(x)Y'+ a0(x)Y = f(x)      *

*         (C) L.M.Berkovich & F.L.Berkovich, 1992             *

*              Samara Computer Algebra Group                  *

*         Samara State University, Samara, RUSSIA             *

* INPUT:         solde(a2,a1,a0,f); don't use E,I,T           *

* For full information please, enter INFO();                  *
```

```
**************************************************************

solde(x^2,0,-a*x+3/16,0);
*Summary of the operations*

**************************************************************

                      2                    16*A*X - 3
*The equation was:   (X )*y''+ (0)*y'+ ( - ------------)*y = 0
                                                16


                                                  16*A*X - 3
*The semi-invariant by dependent variable: J0=  - ------------
                                                          2
                                                      16*X


*The exponential solution doesn't exist

                       2                      16*A*X - 35
*The adjoint equation: X *y'' + (4*X)*y' + ( - -----------)*y=0
                                                    16


*The adjoint equation has not exponential solution

                             1/4                1
*The transformation:   y = (X    )*z,  dt = (---------) dx
                                              SQRT(X)


*  leads to    z''(t)+(0)*z'(t)+ ( - A)*z(t) = 0

*The factorization:

             4*SQRT(A)*X + SQRT(X)        4*SQRT(A)*X + SQRT(X)
*L = (D - (- ---------------------))(D - (---------------------))
                    4*SQRT(X)*X                  4*SQRT(X)*X

*Fundamental system of solutions of Ly=0:
                                                       1/4
        1/4  2*SQRT(X)*SQRT(A)                        X
   Y1= X    *E                            Y2=--------------------
                                                2*SQRT(X)*SQRT(A)
                                               E

*************************************************************
```

Special case of this equation with a=1 was solved by Kovačić [**1**] in a different way.

The KAMKE's collection of integrable equations was essentially extended by D. S. MITRINOVIĆ [**8**] (see also the papers of different authors published in [**16**]). The previous version of the program is described in [**17**]. Semi-invariant with respect to the transformation of independent variable (12) was considered by T. PEJOVIĆ [**14**].

The program version 1.0 was demonstrated at the IV International Conference on Computer Algebra in the Physical Research, Dubna, USSR, May 22–26, 1990. Description of the algorithm was published in [**17-19**]. The version 1.1 (L. M. BERKOVICH and M. L. NECHAEVSKY) which is an improvement of 1.0 was demonstrated on ISSAC'91, Bonn (unpublished). The given version provides a significant extension of the possibilities of the previous ones.

Acknowledgement. The authors are grateful to Dr. V .T. MARKOV and to Mrs L. A. KOZHEVNIKOVA for their help in preparing the paper for publication.

REFERENCES

1. J. KOVAČIĆ: *An Algorithm for Solving Second Order Linear Homogeneous Differential equations.* J. Symb. Comp., **2**, 2 (1986).
2. F. SHWARZ: *A factorization algorithm for linear ordinary differential equations.* Proc. of ISSAC'89, ACM (1989), 17–25.
3. L. M. BERKOVICH: *Factorization and Transformations of Ordinary Differential Equations.* Saratov Univ. Publishers, Saratov, 1989 (in Russian).
4. L. M. BERKOVICH, F. L. BERKOVICH: *Algorithm of Solving Second Order Linear Ordinary Differential Equations and its Implementation in* REDUCE. Theory and Applications in Computer Algebra. RIMS Kokyuoku 848, Sept. 1993, Kyoto Univ., Kyoto, Japan, pp. 100–108.
5. L. M. BERKOVICH, F. L. BERKOVICH: SOLDE, A REDUCE *Package for Solving of Second Order Linear Ordinary Differential Equations.* Proceedings of the Russian Colloquium "Modern Group Analysis and Problems of Mathematical Modelling", Samara, 7–11 June 1993, Samara Univer. Publishers, Samara, 1993, 38–45.
6. E. KAMKE: *Differentialgeichungen. Lösungmethoden und Lösungen.* Leipzig, 1959.
7. G. MURPHY: *Ordinary Differential Equations.* New York, 1960.
8. D. S. MITRINOVIĆ: *Supplement to russian editions of* [**6**]. Moscow, Nauka, 1967 (ed. N. Kh. Rozov).
9. M. F. SINGER: *Formal Solutions of Differential Equations.* J. Symb. Comp., **10** (1990), 59–94.
10. A. C. HEARN: REDUCE *User's manual.* Rand. Corp., Santa Monica, 1987.

11. O. Borůvka: *Lineare Differential Transformationen 2. Ordnung.* Berlin, 1967.
12. L. M. Berkovich: *Transformations of Linear Ordinary Differential Equations.* Kuibyshev State Univ, 1978 (in Russian).
13. G. Mammana: *Sopra un nuovo metodo di studio delle equazioni differeziali lineari.* Math. Z., **25** (1926), 734–748.
14. T. Pejović: *Diferencijalne Jednačine.* Beograd, 1965.
15. L. M. Berkovich: *Second order related lineare ordinare differential equations.* Different. Eq., Minsk, **25**, 2 (1989), 199–201 (in Russian).
16. Univ. Beograd., Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 1956–1983.
17. L. M. Berkovich, V. P. Gerdt, Z. T. Kostova, M. L. Nechaevsky: *Second Order Reducible Linear Differential Equations.* Preprint, JINR E5–89–141, Dubna, 1989.
18. L. M. Berkovich, V. P. Gerdt, Z. T. Kostova, M. L. Nechaevsky: *Integration of some classes of linear ordinary differential equations.* Comp. Algebra in Physical Research, World Scientific, Singapore, 1991, 350–356.
19. L. M. Berkovich, V. P. Gerdt, Z. T. Kostova, M. L. Nechaevsky: *Second order reducible linear differential equations, Packags of applied programs.* Software of math. model. Moscow, Nauka, Comp. Centre of USSR Acad.Sci., 1991, 9–24.

Department of Algebra and Geometry,
Samara State University,
443011, Samara, Russia

(Received August 1, 1994)
(Revised May 22, 1995)

Department of Applied Mathematics,
Samara State Aerospace University,
443086, Samara, Russia