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SOME REMARKS ON GRAPH EQUATION $G^2 = \overline{G}$

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We investigate solutions to the graph equation from the title, or more precisely, we investigate those graphs whose square and complement are equal (i.e. isomorphic). Although this equation looks to be simple, it happens to be very hard (when attempting to find a general solution). Here we only offer some interesting observations and/or solutions.

0. INTRODUCTION

In this paper we follow the terminology and notation from [**7**]. Here we give some necessary notation and/or definitions which will be used throughout the paper:

$\Delta(G)$ - maximal degree of a graph G;

$\mathrm{ecc}(v; G)$ - eccentricity of a vertex v in a graph G;

$d(u, v; G)$ - distance between vertices u and v in a graph G.

For a vertex $v \in V\ (= V(G))$, $V_0(v) \cup V_1(v) \cup \cdots \cup V_k(v)$ is the *distance partition* of V induced by v, i.e. the partition where $V_i(v) = \{u | d(u, v; G) = i\}$, $i = 0, 1, \ldots, k$, where $k = \mathrm{ecc}(v; G)$; in particular, $V_0(v) = \{v\}$.

$G[D]$ is the *distance graph* of G (D is a set of positive integers), i.e. the graph with the same vertex set as G with two vertices u and v adjacent if and

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only if $d(u,v;G) \in D$. In particular, $G^2 = G[\{1,2\}]$, $G^n = G[\{1,2,\dots,n\}]$, $\overline{G} = G[\{2,3,\dots\}]$, etc.

mG is the *m-fold graph* of G ($m = (m_1,\dots,m_n)$ is an n-tuple of non-negative integers, $n = |V(G)|$), i.e. the graph obtained from G as follows:

(i) to each vertex $v_i \in V(G)$, there corresponds m_i vertices in mG, namely $(v_i,1),\dots,(v_i,m_i)$;

(ii) two vertices (u,i) and (v,j) are adjacent in mG if and only if u and v are adjacent in G.

There are many problems in graph theory that can be stated in the form of graph equations (an exhaustive list of such problems, collected until the end of 70-ties, can be found in [**4**]). The graph equation

$$G^2 = \overline{G} \tag{1}$$

to our knowledge, was first posed by S. Schuster [**8**]. Some attepmts were made by M. Capobianco et al. (see [**2, 3**]) to solve it, but except some basic observations and/or particular solutions very little progress has been done. It seems that Eqn. (1), similarly to the equation $G = \overline{G}$ (whose solutions are all self-complementary graphs), cannot be solved in a suitable form. Having this in mind we shall not pursue here to find (or to describe) the whole solution set of Eqn. (1); rather, we shall try to construct some new solutions and/or to give some techiques for producing new solutions from the existing ones. We conclude the paper with some questions interesting to be examined.

1. PRELIMINARY OBSERVATIONS

In this section we first prove several facts which describe to some extent the structure of graphs satisfying Eqn. (1). Since many of these observations also apply to the following (more general) equation

$$G^n = \overline{G} \tag{2}$$

we shall focus our attention to it. In what follows, if not otherwise told, G denotes the non-trivial solution to Eqn. (2), i.e. a solution $G \neq K_1$.

Lemma 1.1. *G is connected, and* $\operatorname{ecc}(v;G) \in \{n+1,\dots,2n\}$ *for any vertex v.*

Proof. Assume G is disconnected. Then G^n is disconnected, while $\overline{G}$ is connected; a contradiction. Next assume that for some vertex of G, say v, $\operatorname{ecc}(v;G) \le n$. But then G $(= \overline{G^n})$ is disconnected; again a contradiction. Since $n \ge 2$, $\operatorname{ecc}(u;G) \ge 3$ and thus $\operatorname{ecc}(u;\overline{G}) = 2$ for any vertex u. Now suppose that $\operatorname{ecc}(v;G) = k$ for some vertex v. Then clearly $\operatorname{ecc}(v;G^n) = \lfloor \frac{k+n-1}{n} \rfloor$. Consequently, $k \in \{n+1,\dots,2n\}$, as required. $\square$

Lemma 1.2. *If v is a vertex of G with maximal degree, then* $\mathrm{ecc}\,(v;G) = n+1$.

Proof. Suppose to the contrary, and let v be a vertex of G with maximal degree such that $\mathrm{ecc}(v;G) > n+1$. Then any vertex u such that $d(u,v;G) > n+1$ is adjacent in $\overline{G^n}$ to v and all its neighbours. But then there is a vertex in $G(= \overline{G^n})$ whose degree is larger than the degree of v. $\square$

Corollary 1.3. *If G is regular, then $ecc(v;G) = n+1$ for any vertex v, i.e. G is self-centered.*

Lemma 1.4. *If* $\Delta\ (= \Delta(G))$ *is the maximal (vertex) degree of G, then*

$$|V(G)| \leq \frac{\Delta(\Delta-1)^n\Delta^2 - 2\Delta - 2}{\Delta - 2}. \tag{3}$$

Proof. By Lemma 1.2 there is a vertex, say v, such that $\mathrm{ecc}(v;G) = n+1$. Let $V_0(v) \cup V_1(v) \cup \cdots \cup V_{n+1}(v)$ be the distance partition of $V(G)$ induced by v. Then $|V_1(v)| \leq \Delta$, $|V_{i+1}(v)| \leq (\Delta-1)|V_i(v)|$ for $i = 1, \ldots n-1$, and $|V_{n+1}(v)| \leq \Delta$ (since v is adjacent to each vertex from $V_{n+1}(v)$ in $\overline{G^n}$). Now (3) follows at once. $\square$

The next lemma applies only to graphs G satisfying Eqn. (1).

Lemma 1.5. *If G is a solution to Eqn.* (1), *then G is a block.*

Proof. Assume that v is a cut-vertex. Then $\mathrm{ecc}(v;G) = 3$, since otherwise there exists a vertex, say u, such that $\mathrm{ecc}(u;G) = 5$. Now let $V_0(v) \cup V_1(v) \cup V_2(v) \cup V_3(v)$ be the distance partition of $V(G)$ induced by v. Also, since v is a cut-vertex, let $V_1(v) = V_1(v)^{'} \cup V_1(v)^{''}$ where $V_1(v)^{'}$ and $V_1(v)^{''}$ belong to different components in $G - v$. Also assume that the vertices from $V_1(v)^{''}$ belong to the same component as those from $V_2(v)$; thus the vertices from $V_1(x)^{'}$ and those from $V_2(v)$ are at distance 3 in G. Consider now $\overline{G^2}$, and the partition of its vertex which stems from the above partition of the vertex set of G. We then have: v is adjacent to all vertices from $V_3(v)$, but no others; the vertices from $V_1(v)$ are mutually non-adjacent; any vertex from $V_1(v)^{'}$ is adjacent to all vertices from $V_2(v) \cup V_3(v)$; any vertex from $V_1(v)^{''}$ is adjacent to at least one vertex from $V_2(v) \cup V_3(v)$ (since $\overline{G^2}$ need to be connected). With the above observations, $\mathrm{ecc}(w;\overline{G^2}) = 2$ for any vertex $w \in V_1(v)^{'}$. This contradicts Lemma 1.1, and hence G is a block. $\square$

Remark 1.6. From the above lemma it follows that G cannot be a tree. In [**3**] it was proved that G must be a $D(1,3)$-balanced graph (for definition see [**3**]) and that among trees there are only three such non-trivial trees, none of which is a solution to Eqn. (1).

2. SOME SOLUTIONS

We first investigate the (non-trivial) solutions of Eqn. (1).

By exhaustive search (carried on by the programming package GRAPH, see [**5, 6**]) it was established that among the graphs up to 7 vertices the only (non-trivial) solution is the graph $G = C_7$ (which was also registered in [**2**]). In addition, among the graphs on 8 vertices, only one further solution exists: namely the graph obtained from C_7 by adding a vertex which is adjacent to two vertices from a cycle at distance 2. This construction, as observed by F. Buckley (according to [**2**]), can be extended by adding any number of nonadjacent vertices which are adjacent to two fixed vertices from a cycle at distance 2 (see Fig. 1).

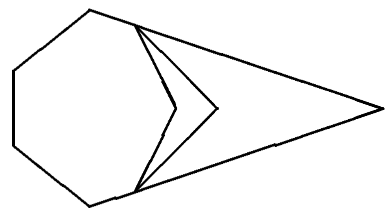

Fig. 1

Now we provide a (more) general technique for producing solutions to Eqn. (1). For convenience, we shall put it in the equivalent form

$$f(G) = G, \tag{4}$$

where $f(\cdot) = \overline{(\cdot)^2}$.

We assume that the vertex sets of graphs G and $f(G)$ from Eqn. (2) are both equal to $\{1, \ldots, n\}$. Let a permutation $\pi \in S_n$ be an isomorphism between G and $f(G)$, i.e.

$$u \sim v \text{ (in } G) \iff \pi(u) \sim \pi(v) \text{ (in } f(G)).$$

Next let $\pi_1 \circ \cdots \circ \pi_k$ $(= \pi)$ be a decomposition of π into disjoint cycles. Let $m = (m_1, \cdots, m_n)$ be an n-tuple of positive integers chosen so that $m_i = m_j$ whenever the vertices i, j belong to the same cycle (i.e. there exist s such that $i, j \in \pi_s$). With these assumptions we have:

Lemma 2.1. *If G is a solution to the Eqn.* (1) *and if mG is obtained as above, then mG is also a solution to the Eqn.* (1).

Proof. Assume $f(G) = G$, and let m be choosen as above. Then we have:

$$(u, i) \sim (v, j) \text{ in } mG \iff (\pi(u), i) \sim (\pi(v), j) \text{ in } f(mG).$$

Notice, since $d((u, i), (u, j)) = 2$ (in mG) for any u whenever $i \neq j$, the vertices such as $(u, i), (u, j)$ (and $(\pi(u), i), (\pi(u), j)$) are always non-adjacent. □

Example 2.2. Let $G = C_7$, and also assume that the vertices of a cycle are numbered in natural order $1, 2, \ldots, 7$ around the cycle. Then, as it is easy to check, two representative isomorphisms between G and $f(G)$ are:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 7 & 3 & 6 & 2 & 5 \end{pmatrix} \ (= (1)(243756)), \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 \end{pmatrix} \ (= (1)(253)(467)).$$

Now by Lemma 2.1 we have that the graph $(p, q, q, r, q, r, r)C_7$ for any positive integers p, q, r is a solution to Eqn. (1) (notice, that the solution which stems from the first permutation is contained as a special case of the second one - just put $q = r$).

Remark 2.3. By the above construction (or F. BUCKLEY's construction which is the special case of the above one) it follows that for any $n \geq 7$ there exists a graph on n vertices being the solutions to Eqn. (1).

We now turn to regular solutions. If G is regular of degree r, then by (3) we have $|V(G)| \leq r^2 + r + 1$. For $r = 2$ we can easily get that $G = C_7$ is the only solution. For $r = 3$ we have $|V(G)| \leq 12$ (since cubic graphs have even number of vertices). By examining the catalogue of all cubic graphs up to 12 vertices (i.e. by programming package GRAPH) we have obtained only one solution (to Eqn. (1)) within this class of graphs. It is depicted in Fig. 2.

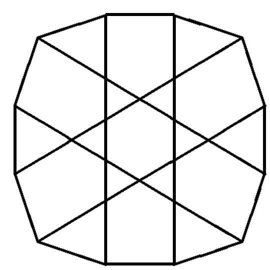

Fig. 2

For $r = 4$ we have $|V(G)| \leq 21$. One solution with 14 vertices can be easily constructed by making use of Lemma 2.1. (namely, $(2, 2, 2, 2, 2, 2, 2)C_7$). According to [**7**] (as noted by J. AKIYAMA [**1**]) the graph $C_{17}[\{1, 4\}]$ is a solution to Eqn. (1). Other solutions (if any) can be found by an exhaustive computer search.

In what follows we add few remarks on Eqn. (2) with $n \geq 3$. Firstly, it is easy to check that $G = C_{2n+3}$ is a solution to Eqn. (2) for any n. Also a construction given in Lemma 2.1 applies again (for any n). In addition, for $n = 3$ we have found that $G = C_{27}[\{1, 5\}]$ is a solution to Eqn. (2).

3. FINAL REMARKS

In this section we only put some open problems.

PROBLEM 1: Is there a solution to Eqn. (1) (clearly non-regular) whose diameter is equal to 4?

PROBLEM 2: Is there a solution to Eqn. (1) having a triangle?

PROBLEM 3: Is there a solution to Eqn. (1) which is regular of degree r having $r^2 + r + 1$ vertices?

In the case of Problem 3 r is clearly even any such graph, if exists, has the girth not less than 5.

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