Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 5 (1994), 29–34.

ON THE DISTANCE OF SOME COMPOUND GRAPHS

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The distance $d(G)$ of a connected graph G is the sum of the distances of all pairs of vertices of G. Let S be a connected graph. The sets I_n and J_n are defined so that $I_1 = J_1 = \{S\}$, and for $n > 1$ the elements of $I_n(J_n)$ are graphs obtained by identifying (joining) a vertex of S with a vertex of an element of $I_{n-1}(J_{n-1})$. It is demonstrated that if all the vertices of S are equivalent, then for $n \geq 1$, $G, G' \in I_n$ implies $d(G) \equiv d(G') \pmod{(|S|-1)^2}$, and $G, G' \in J_n$ implies $d(G) \equiv d(G') \pmod{|S|^2}$; $|S|$ is the number of vertices of S.

1. INTRODUCTION

In this paper we are concerned with finite undirected graphs. Throughout the entire paper it is understood that all the graphs considered are connected. The number of vertices of a graph G is denoted by $|G|$.

The sum $d(G)$ of distances of all pairs of vertices of a graph G, as well as the closely related average vertex distance $\binom{|G|}{2}^{-1} d(G)$, are the topic of numerous contemporary mathematical researches; for some of the most recent works in this field see [**1, 4, 5, 8–12**].

It was noticed some time ago that for certain classes of graphs it is possible to find an integer m, such that the distances of all graphs from this class are congruent modulo m. The simplest such regularity is observed in the class $B_{a,b}$ of connected bipartite graphs with $a + b$ vertices [**2**]: For all $G \in B_{a,b}$, $d(G) \equiv ab \pmod{2}$. Eventually, some less obvious results of this type were discovered [**6, 7**], of which we mention here only the case of chains of polygons. Each polygon in such a chain

[0] 1991 Mathematics Subject Classification: 05C12

has $2k$ vertices, and each two adjacent polygons share one edge. If G and G' are two such chains, consisting of equal number of polygons, then [**7**] $d(G) \equiv d(G') \pmod{m}$ where $m = 2(k-1)^2$.

In this paper we communicate two further results of the same kind. In order to state them we need some preparation.

2. PRELIMINARIES

The vertex set of the graph G is denoted by $V(G)$. The distance between the vertices u and v of G is denoted by $d(u, v|G)$; it is equal to the number of edges in the shortest path that connects u and v [**3**]. The distance $d(u|G)$ of the vertex u of G, and the distance $d(G)$ of the graph G are defined as

$$(1) \qquad d(u|G) := \sum_{v \in V(G)} d(u, v|G), \qquad d(G) := \frac{1}{2} \sum_{u \in V(G)} d(u|G).$$

Let S be a graph. We now recursively define two classes of graphs, $I_n = I_n(S)$ and $J_n = J_n(S)$.

Definition 1. *$I_1 = \{S\}$. If $n > 1$, then every element of I_n is obtained by taking an element of I_{n-1} (which, of course, is a graph) and identifying one of its vertices with a vertex of an additional copy of S. The class I_n consists of all graphs which can be constructed in this manner.*

Definition 2. *$J_1 = \{S\}$. If $n > 1$, then every element of J_n is obtained by taking an element of J_{n-1} (which is a graph) and joining one of its vertices to a vertex of an additional copy of S. The class J_n consists of all graphs which can be constructed in this manner.*

The number of vertices and edges of S are denoted by $|S|$ and $e(S)$, respectively.

All graphs that belong to I_n have $n|S| - n + 1$ vertices and $ne(S)$ edges. All graphs that belong to J_n have $n|S|$ vertices and $ne(S) + n - 1$ edges. We mention in passing that $I_n(K_2)$ as well as $J_n(K_1)$ coincide with the set of $n-$vertex trees; this fact is of little use for us because Theorems 1 and 2 (see below) reduce to trivial statements when $|S| = 2$ and $|S| = 1$, respectively.

In what follows we will be interested in the special cases of the sets $I_n(S)$ and $J_n(S)$ when the graph S has the property π.

Definition 3. *We say that a connected graph S has the property π is for any two vertices u and v of S, $d(u|S) = d(v|S)$.*

Among graphs that possess property π are those whose all vertices are equivalent (i.e., belong to the same orbit of the automorphism group).

3. STATEMENT OF THE RESULTS

Theorem 1. *Let S be a graph with $|S|$ vertices and $m = (|S| - 1)^2$. Let $G, G' \in I_n(S)$, where n is a positive integer. If S has the property π, then $d(G) \equiv d(G') \pmod{m}$.*

Theorem 2. *Let S be a graph with $|S|$ vertices and $m = |S|^2$. Let $G, G' \in J_n(S)$, where n is a positive integer. If S has the property π, then $d(G) \equiv d(G') \pmod{m}$.*

4. PROOF OF THEOREM 1

First observe that because I_1 possesses only one element (S), Theorem 1 holds for $n = 1$ in a trivial manner. (The same is, of course, true for Theorem 2.)

Assume thus $n > 1$ and consider a graph G, $G \in I_n$. Let this graph be obtained from a graph H, $H \in I_{n-1}$, and a copy of S, so that a certain vertex y of H is identified with a certain vertex z of S. Hence, in what follows $y \in V(H)$ and $z \in V(S)$ denote he same vertex of G.

Lemma 1. *Let $G, G' \in I_n(S)$, S has property π, $x \in V(G)$ and $x' \in V(G')$. Then for all $n \geq 1$,*

$$d(x|G) \equiv d(x'|G') \mod (|S| - 1). \tag{2}$$

Proof. Let x be an arbitrary vertex of G. Then either $x \in V(H)$ or $x \in V(S)$ (or both, in which case $x = y = z$).

If $x \in V(H)$, then by taking into account (1) we have

$$d(x|G) = \sum_{u \in V(H)} d(x, u|H) + \sum_{u \in V(S)} d(x, u|G) - d(x, y|H). \tag{3}$$

Because of

$$d(x, u|G) = d(x, y|H) + d(z, u|S) \tag{4}$$

Eq. (3) is transformed into

$$d(x|G) = d(x|H) + d(z|S) + (|S| - 1)d(x, y|H). \tag{5}$$

Let G' be another graph from I_n and let its vertices and fragments be labeled analogously as in the graph G. Then from (5),

$$\begin{aligned} d(x|G) - d(x'|G') &= d(x|H) - d(x'|H') + d(z|S) - d(z'|S) \\ &\quad + (|S| - 1)[d(x, y|H) - d(x', y'|H')]. \end{aligned} \tag{6}$$

If S has property π, then the value of $d(z|S)$ is independent of the choice of the vertex z, and (6) is simplified:

$$d(x|G) - d(x'|G') = d(x|H) - d(x'|H') + (|S| - 1)[d(x, y|H) - d(x', y'|H')]. \tag{7}$$

In the above formulas, of course, $H' \in I_{n-1}$.

If $n = 1$, then $G = G' = S$. Because of property π, $d(x|G) = d(x'|G')$, and therefore $d(x|G)$ and $d(x'|G')$ are certainly congruent modulo $|S| - 1$.

If $n = 2$, then $H = H' = S$. Because of property π, $d(x|H) = d(x'|H')$. Then from (7) we see that, again, $d(x|G)$ and $d(x'|G')$ are congruent modulo $|S| - 1$.

Using (7) we now verify by induction on n that $d(x|G)$ and $d(x'|G')$ are congruent modulo $|S|-1$ for all values of n, $n \geq 1$.

Hence, Eq. (2) holds in the case $x \in V(H)$.

Examine now the other possible case, namely $x \in V(S)$. Then in parallel to (3) and (4) one has

$$d(x|G) = \sum_{u \in V(H)} d(x,u|G) + \sum_{u \in V(S)} d(x,u|S) - d(x,z|S)$$
$$d(x,u|G) = d(x,z|S) + d(y,u|H)$$

which result in

$$d(x|G) = d(y|H) + d(x|S) + (|H|-1)d(x,z|S). \tag{8}$$

Utilizing the facts that $d(x|S)$ is independent of x, and that $|H| = (n-1)|S| - n + 2$, we obtain in analogy to (7):

$$d(x|G) - d(x'|G') = d(y|H) - d(y'|H') + (n-1)(|S|-1)[d(x,z|S) - d(x',z'|S)]. \tag{9}$$

The same reasoning as in the case of Eq. (7) leads now to the conclusion that Eq. (2) holds for $x \in V(S)$.

By this the proof of Lemma 1 is completed.

Proof of Theorem 1. As already explained, Theorem 1 needs to be verified only for $n > 1$. From the definition of graph distance, and bearing in mind the structure of the graph $G \in I_n$, we immediately have

$$d(G) = d(H) + d(S) + \sum_{u \in V'(H)} \sum_{v \in V'(S)} d(u,v|G) \tag{10}$$

where $V'(H) = V(H) \backslash \{y\}$ and $V'(S) = V(S) \backslash \{z\}$. For the vertices u, v, specified in Eq. (10),

$$d(u,v|G) = d(u,y|H) + d(z,v|S). \tag{11}$$

Substituting (11) back into (10) and taking into account that $|V'(H)| = |H| - 1 = (n-1)(|S|-1)$ and $|V'(S)| = |S| - 1$, we obtain

$$d(G) = d(H) + d(S) + (|S|-1)d(y|H) + (n-1)(|S|-1)d(z|S). \tag{12}$$

Assuming that S has property π and using the same notation as in the proof of Lemma 1, we obtain from (12),

$$d(G) - d(G') = d(H) - d(H') + (|S|-1)[d(y|H) - d(y'|H')]. \tag{13}$$

For $n = 2$, $d(H) = d(H')$ and $d(y|H) = d(y'|H')$. Therefore $d(G)$ and $d(G')$ coincide and therefore their difference is divisible by $m = (|S|-1)^2$. Because of Lemma 1, the last term on the right–hand side of (13) is divisible by m for all values of n. Therefore, $d(G)$ and $d(G')$ are congruent modulo m if and only if $d(H)$ and $d(H')$ are congruent modulo m.

Theorem 1 is now deduced from (13) by means of a simple inductive argument.

5. PROOF OF THEOREM 2

Theorem 2 can be verified analogously as Theorem 1. In fact, the proof is somewhat simpler because no vertex of a graph $G \in J_n$ belongs simultaneously to both the fragments $H \in J_{n-1}$ and S. As before, it may be assumed that $n > 1$. Consider a graph G, $G \in J_n$. Let this graph be obtained from a graph H, $H \in J_{n-1}$, and a copy of S, so that a new edge is introduced between a vertex y of H a vertex z of S. In this case, of course, y and z are distinct vertices of G.

As before, we first establish a congruence relation for the vertex distances.

Lemma 2. *Let $G, G' \in J_n(S)$, S has property π, $x \in V(G)$ and $x' \in V(G')$. Then for all $n \geq 1$, $d(x|G) \equiv d(x'|G') \pmod{|S|}$.*

Sketch of the proof of Lemma 2. If $x \in V(H)$, then in parallel to (5) and (7),

$$d(x|G) = d(x|H) + d(z|S) + |S|\,[d(x, y|H) + 1]$$

and

$$d(x|G) - d(x'|G') = d(x|H) - d(x'|H') + |S|\,[d(x, y|H) - d(x', y'|H')].$$

If $\in V(S)$, then instead of (8) and (9) one has

$$d(x|G) = d(y|H) + d(x|S) + |H|\,[d(x, z|S) + 1]$$

and

$$d(x|G) - d(x'|G') = d(y|H) - d(y'|H') + (n-1)|S|\,[d(x, z|S) - d(x', z'|S)]$$

where $|H| = (n-1)|S|$. In both cases, Lemma 2 is readily verified by induction on the number n of S–fragments in the graph G.

Sketch of the proof of Theorem 2. Instead of Eqs. (10)–(13) we now arrive at

$$\begin{aligned}
&d(G) = d(H) + d(S) + \sum_{u \in V(H)} \sum_{v \in V(S)} d(u, v|G),\\
&d(u, v|G) = d(u, y|H) + d(z, v|S) + 1,\\
&d(G) = d(H) + d(S) + |S|d(y|H) + (n-1)|S|d(z|S) + (n-1)|S|^2,\\
&d(G) - d(G') = d(H) - d(H') + |S|[d(y|H) - d(y'|H')].
\end{aligned}$$

Theorem 2 follows by means of Lemma 2, using induction on the parameter n.

Acknowledgement. The author thanks the Mathematical Institute in Belgrade for financial support of this research.

REFERENCES

1. I. ALTHÖFER: *Average distance in undirected graphs and the removal of vertices.* J. Combin. Theory, B 48 (1990), 140–142.
2. D. BONCHEV, I. GUTMAN, O. E. POLANSKY: *Parity of the distance numbers and Wiener numbers of bipartite graphs.* Commun. Math. Chem. **22** (1987), 209–214.
3. F. BUCKLEY, F. HARARY: *Distance in Graphs.* Addison–Wesley, Reading 1990.
4. P. DANKELMANN: *Computing the average distance of an interval graph.* Inform. Process. Lett. **48** (1993), 311–314.
5. P. DANKELMANN: *Average distance and independence number.* Discrete Appl. Math. **51** (1994), 75–83.
6. I. GUTMAN: *Wiener numbers of benzenoid hydrocarbons: two theorems.* Chem. Phys. Letters **136** (1987), 134–136.
7. I. GUTMAN: *On distance in some bipartite graphs.* Publ. Inst. Math. (Beograd) **43** (1988), 3–8.
8. I. GUTMAN, Y. N. YEH: *The sum of all distances in bipartite graphs.* Math. Slovaca **44** (1994), (to appear).
9. I. GUTMAN, Y. N. YEH, J. C. CHEN: *On the sum of all distance in graphs.* Tamkang Math. J. **25** (1994), 83–86.
10. B. MOHAR: *Eigenvalues, diameter and mean distance in graphs.* Graphs and Combinatorics **7** (1991), 53–64.
11. L. ŠOLTÉS: *Transmission in graphs: a bound and vertex removing.* Math. Slovaca **41** (1991), 11–16.
12. P. WINKLER: *Mean distance in a tree.* Discrete Appl. Math. **27** (1990), 179–185.

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(Received September 30, 1994)