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ON A SOLVABLE CLASS OF N^{th} ORDER LINEAR DIFFERENTIAL EQUATIONS

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Differential equations (2) and (5) are solved.

It is a known result that if one has a solution $y = f(x)$ to the linear homogeneous D. E. (differential equation) $[D^n + u_1(x)D^{n-1} + \cdots + u_n(x)]y = 0$, one can depress the order by one by means of the substitution $y = zf(x)$. Thus for the case $n = 2$, one solution leads to the general solution since the resulting 1^{st} order linear D. E. can always be solved.

Here we exhibit a class of n^{th} order D.E.'s whose general solution will follow from the knowledge of one solution. The motivation for this note arises from the problem of proving the identity

$$2^{2n+1}[D^n x^{n+1/2} D^{n+1}]e^{\sqrt{x}} = e^{\sqrt{x}}. \tag{1}$$

I had first seen this problem a long time ago in one of the many problem books of D. S. Mitrinović and had proved it at that time. More recently in preparing a talk on the 100th anniversary of the American Mathematical Monthly problem section, I fond it again as a proposed problem in the Monthly's first volume [**1**]. There were two inductive proofs given, one of which was not really complete. Since I wanted to see a simple proof than the ones given, I tried to recall my original proof but without success. This led me to solve the D.E.

$$[D^n x^{n+1/2} D^{n+1}]y = y \tag{2}$$

and generalizations thereof (the factor 2^{2n+1} was removed by letting $x \to 4x$ so that $e^{\sqrt{x}} \to e^{2\sqrt{x}}$). The knowledge of the subsequent solution of (2) leads to a very easy way of obtaining all the solutions of (2) from just knowing that $e^{2\sqrt{x}}$ is one solution. By letting $x \to k^2 t$ where $k^{2n+1} = 1$, (2) remains the same. Hence the general solution of (2) is

$$y = \sum_m A_m e^{2\omega^m \sqrt{x}}, \tag{3}$$

where ω is a primitive $(2n+1)$-root of unity.

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To obtain the solution of (2) ab inito, we note that it is somewhat like an EULER linear D.E., so we make the substitution $x = e^z$ and use the exponential shift theorem $e^{az}L(D) \equiv L(D-a)e^{az}$ to obtain

$$\left(D-\frac{1}{2}\right)\left(D-\frac{3}{2}\right)\cdots\left(D-n+\frac{1}{2}\right)D(D-1)\cdots(D-n)y = e^{(n+1/2)z}y.$$

To get rid of the fractions, we let $z = 2s$: $D(D-1)(D-2)\cdots(D-2n-1)y = 2^{2n+1}e^{(2n+1)s}y$. Now letting $t = e^s$, we get $D^{2n+1}y = 2^{2n+1}y$ so that

$$y = \sum_m A_m e^{2\omega^m t} = \sum_m A_m e^{2\omega^m\sqrt{x}}.$$

Combining the substitutions that were made, we obtain the known operator identity

$$D^n x^{n+1/2} D^{n+1} = [\sqrt{x}D]^{2n+1}. \tag{4}$$

If I was aware of this identity at the time, I would not have been led to the more general D.E.

$$D^n[x^{n+(r-1)/r}D^{n+1}]^{r-1}y = y \tag{5}$$

since it leads to an immediate solution of (2). Solving (5) in the same manner as for (2), the general solution is given by

$$y = \sum_m A_m \exp(r\omega^m x^{1/r}). \tag{6}$$

Also knowing one solution $y = e^{rx^{1/r}}$ of (5), we could have obtained the general solution as before. Finally as before, the solution (6) leads to the following generalization of operator identity (4):

$$D^n[x^{n+(r-1)/r}D^{n+1}]^{r-1} \equiv [x^{(r-1)/r}D]^{rn+r-1}. \tag{7}$$

Postscript: I subsequently came across identity (1) again in [**2**], p. 86. His proof is gotten by expanding $e^{\sqrt{x}}$ into a power series and carrying out the indicated differentiations. Since this book was first published in 1885, it is quite likeky that (1) appeared as a problem in a Cambridge University examination paper.

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