UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 4 (1993), 43–48.

ON A SUM INVOLVING THE NUMBER OF PRIME FACTORS OF AN INTEGER

*Aleksandar Ivić**

A sharp asymptotic formula for the summatory function of $(1+\Omega(n)/\omega(n))^{\omega(n)}$ is derived. As usual $\omega(n)$ is the number of distinct prime factors of n, and $\Omega(n)$ is the total number of prime factors of n.

During my stay at the Tata Institute in 1990 DR. S. SRINIVASAN asked me to evaluate asymptotically the sum

$$F(x) := \sum_{2\le n\le x}\left(1+\frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)}. \tag{1}$$

Here, as usual, $\omega(n)$ and $\Omega(n)$ denote the number of distinct prime factors of n and the total number of prime factors of n, respectively. At the first glance the sum in (1) seems somewhat bizarre. However, its arithmetic significance comes from the fact that

$$d(n) \le \left(1+\frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} \quad (n>1), \tag{2}$$

where $d(n)$ is the number of divisors of n. Namely, by using the inequality for the arithmetic-geometric means one obtains

$$(\alpha_1+1)\cdots(\alpha_r+1) \le \left(\frac{(\alpha_1+1)\cdots(\alpha_r+1)}{r}\right)^r \quad (\alpha_i>0). \tag{3}$$

Hence if $n = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is the canonical decomposition of n into prime powers, we obtain (2) from (3), and equality holds in (2) if and only if n is a power of a squarefree number. It seems interesting to investigate how much, on the average, one loses in applying (2), and this is how the sum $F(x)$ arises. Since

---

1991 Mathematics Subject Classification: 11N37

*Research financed by the Mathematical Institute, Belgrade

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + O(x^{\frac12}), \tag{4}$$

where $\gamma = 0.577...$ is EULER'S constant, we obtain trivially from (2) and (4) that

$$F(x) \ge x\log x \tag{5}$$

for sufficiently large x. It turns out that the right-hand side of (5) is by a constant factor smaller than the true order of magnitude of $F(x)$, since

$$F(x) \sim C\,x\log x \quad (x\to\infty,\ C>1). \tag{6}$$

The asymptotic formula (6) follows from a much stronger result. Namely, we shall prove the following

Theorem. *Let M be an arbitrary, but fixed natural number. Then there exist constants $A_1, A_2, ..., A_M$ which may be effectively computed such that*

$$\sum_{2\le n\le x}\left(1+\frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} = H(1)\,x\,\log x + \sum_{j=1}^{M}\frac{A_j}{(\log\log x)^j}x\,\log x + O\left(\frac{x\,\log x}{(\log\log x)^{M+1}}\right), \tag{7}$$

where

$$H(s) = \prod_p\left(1-p^{-s}\right)^2\left(1+\frac{2}{p^s - e^{\frac12}}\right) \quad (Re\,s > \frac{1}{\log 4}).$$

It is easily seen that $H(1) > 1$, and from (7) one trivially obtains (6) with $C = H(1)$. We begin the proof of (7) by decomposing the sum $F(x)$ as

$$F(x) = S_1 + S_2 + S_3 + O(x), \tag{8}$$

say, where in S_1 we have $\Omega(n) - \omega(n) \le \sqrt{\omega(n)}$, in S_2 we have $\sqrt{\omega(n)} < \Omega(n) - \omega(n) \le \delta\omega(n)$ for a small, fixed $\delta > 0$, and in S_3 we have $\Omega(n) - \omega(n) > \delta\omega(n)$. This splitting makes sense if $\omega(n) \ge \delta^{-2}$, and the contribution of n for which $\omega(n) < \delta^{-2}$ is easily seen to be $O(x)$. It will turn out that the main contribution to $F(x)$ comes from S_1, while S_2 and S_3 are of a smaller order of magnitude. We shall show that, for some $\eta = \eta(\delta)$ satisfying $\eta < 1$, we have

$$S_3 \ll x\,\log^\eta x. \tag{9}$$

To accomplish this note that, for $x \ge 1+\delta$ and $\delta \ge 0$, we have

$$\log(1+x) \le \frac{\log(2+\delta)}{1+\delta}x. \tag{10}$$

Namely, setting $g(x) := \frac{\log(1+x)}{x}$ it is seen that

$$g'(x) = \frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2} = \frac{x - (1+x)\log(1+x)}{x^2(1+x)} < 0$$

for $x > 0$. Hence $g(x)$ is decreasing for $x > 0$, and (10) follows. In S_3 we have $\Omega(n)/\omega(n) > 1 + \delta$, so that (10) yields

$$\left(1 + \frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} \le (2+\delta)^{\Omega(n)/(1+\delta)}.$$

Note that $(2+\delta)^{1/(1+\delta)} < 2$, so that we obtain

$$S_3 \le \sum_{n \le x} (2+\delta)^{\frac{\Omega(n)}{(1+\delta)}} \ll x \log^{\eta} x, \ \eta = (2+\delta)^{\frac{1}{1+\delta}} - 1 < 1. \tag{11}$$

Here we used the well-known result that $\sum_{n \le x} a^{\Omega(n)} \ll x \log^{Re\, a - 1} x$ if a is a constant such that $|a| < 2$. The proof of this bound follows e. g. by the method of A. Selberg [**3**]. Also, as usual, $f(x) \ll g(x)$ (same as $f(x) = O(g(x))$) means that $|f(x)| < Cg(x)$ for $x \ge x_0$, $g(x) > 0$ and some constant $C > 0$.

Next we shall bound S_2. To do this we need a bound which is a consequence of an asymptotic formula which will also be needed later. This is contained in the following

Lemma. *Let c, d be real numbers such that $c > 0$ and $0 \le d < 2$, and r, k integers such that $r \ge 0$, $k \ge 0$. Let*

$$G(x) = G(x; c, d, r, k) := \sum_{2 \le n \le x} c^{\omega(n)} d^{\Omega(n) - \omega(n)} (\Omega(n) - \omega(n))^r \omega^{-k}(n).$$

Then

$$G(x) = x \log^{c-1} x \left\{ \frac{A_1}{(\log\log x)^k} + \cdots + \frac{A_M}{(\log\log x)^{k+M-1}} + O\left(\frac{1}{(\log\log x)^{k+M}}\right)\right\} \tag{12}$$

for any arbitrary, but fixed integer $M \ge 1$ and effectively computable constants $A_1, \ldots, A_M$ which depend on c, d, r and k.

Proof. The proof follows by the method of [**2**]. The basic principle is that $z^{h(n)}$ is a multiplicative function of n for $z \in C$ if $h(n)$ is an additive arithmetic function. One considers first

$$S(x; z, w) := \sum_{2 \le n \le x} c^{\omega(n)} z^{\omega(n)} w^{\Omega(n) - \omega(n)},$$

where z and w are complex variables satisfying $|z| \le 2c,\ |w| \le 2-\epsilon$ for some $\epsilon > 0$. The reason for the restriction on w, as well as $0 \le d < 2$,is that the generating DIRICHLET series

$$\sum_{n=1}^{\infty} c^{\omega(n)} z^{\omega(n)} w^{\Omega(n)-\omega(n)} n^{-s} = \prod_p \left(1 + czp^{-s} + czwp^{-2s} + czw^2p^{-3s} + \cdots\right)$$

for $\mathrm{Re}\, s > 1$ is absolutely convergent only if $|w| < 2$. Analogously to the formula on p. 41 of [2] one obtains $S(x;z,w) = x \sum_{j=1}^{N} f_j(z,w) \log^{cz-j} x + R_N(x;z,w)$ for any fixed integer $N \ge 1$ and certain regular functions $f_j(z,w)$, which may be written down explicitly. The function $R_N(x;z,w)$ is also regular and satisfies $R_N(x;z,w) \ll x(\log x)^{cRe\, z-N-1}$ uniformly for $|z| \le 2c,\ \ |w| \le 2-\epsilon$. We have

$$\begin{aligned} T_r(x;z) &:= \sum_{2\le n\le x} c^{\omega(n)} z^{\omega(n)} d^{\Omega(n)-\omega(n)} \left(\Omega(n)-\omega(n)\right)^r \\ &= \frac{\partial}{\partial w}\left(\underbrace{w \cdots \left(w \frac{\partial S(x;z,w)}{\partial w}\right)\cdots}_{r\ \text{times}}\right)\Bigg|_{w=d}, \end{aligned}$$

so that $T_r(x;z)$ may be evaluated by using the asymptotic formula for $S(x;z,w)$. To introduce the reciprocals of $\omega(n)$ in the sums defining $T_r(x;z)$ we divide $T_r(x;z)$ by z and integrate over z, from $\epsilon(x)$ to z, where $\epsilon(x) = \log^{-A} x$ with a suitable constant $A > 0$. This will introduce the factor $1/\omega(n)$ in the corresponding asymptotic formula. This procedure, described in detail in the monograph [**1**], is repeated k times, only the last time integration will be from $z = \epsilon(x)$ to $z = 1$. In this way the asymptotic formula (12) will be obtained.

With the asymptotic formula (12) at our disposal we may proceed with the estimation of S_2. Write

$$\begin{aligned} \left(1 + \frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} &= 2^{\omega(n)} \left(1 + \frac{\Omega(n)-\omega(n)}{2\omega(n)}\right)^{\omega(n)} \\ &= 2^{\omega(n)} \exp\left\{\omega(n) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{\Omega(n)-\omega(n)}{2\omega(n)}\right)^k\right\} \qquad (13) \\ &= 2^{\omega(n)} e^{\frac12(\Omega(n)-\omega(n))} \exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{2^k k} \frac{(\Omega(n)-\omega(n))^k}{\omega^{k-1}(n)}\right\}. \end{aligned}$$

Recalling that $0 \le \Omega(n) - \omega(n) \le \delta\omega(n)$ in S_2, we have

$$\exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{2^k k} \frac{(\Omega(n)-\omega(n))^k}{\omega^{k-1}(n)}\right\} \le \exp\left(\delta\left(\Omega(n)-\omega(n)\right)\right).$$

Therefore by using (12), with $d = e^{\frac{1}{2}+\delta}$ and δ sufficiently small, we obtain

$$(14)\qquad S_2 \le \sum_{2\le n\le x,\,\Omega(n)-\omega(n)>\sqrt{\omega(n)}} 2^{\omega(n)} d^{\Omega(n)-\omega(n)}$$

$$\le \sum_{2\le n\le x} 2^{\omega(n)} d^{\Omega(n)-\omega(n)} \frac{(\Omega(n)-\omega(n))^{6M}}{\omega^{3M}(n)} \ll \frac{x\log x}{(\log\log x)^{3M}}$$

for any fixed integer $M \ge 1$, so that the contribution of S_2 is absorbed in the error term in (7).

To evaluate S_1 we use (13), noting that for any fixed integer $N \ge 2$

$$\exp\left\{\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{2^k k} \frac{(\Omega(n)-\omega(n))^k}{\omega^{k-1}(n)}\right\}$$

$$= \prod_{k=2}^{N} \exp\left\{\frac{(-1)^{k-1}}{2^k k} \frac{(\Omega(n)-\omega(n))^k}{\omega^{k-1}(n)}\right\} \exp\left\{O\left(\frac{(\Omega(n)-\omega(n))^{N+1}}{\omega^N(n)}\right)\right\}.$$

In S_1 we have $0 \le \Omega(n)-\omega(n) \le \sqrt{\omega(n)}$, which implies that

$$\frac{(\Omega(n)-\omega(n))^k}{\omega^{k-1}(n)} \le 1 \quad (k \ge 2),$$

so that we may use the expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!} + O\left(e^{|x|}|x|^{N+1}\right) \quad (|x| \le 1)$$

for each exponential factor in the above product. Thus we shall obtain, for n in S_1,

$$(15)\qquad \left(1+\frac{\Omega(n)}{\omega(n)}\right)^{\omega(n)} = 2^{\omega(n)} e^{\frac{1}{2}(\Omega(n)-\omega(n))} \times$$

$$\times \left\{1 + \sum_{k=1}^{N}\sum_{r=k+1}^{2k} d_{r,k} \frac{(\Omega(n)-\omega(n))^r}{\omega^k(n)} + O\left(e^{\delta(\Omega(n)-\omega(n))} \frac{(\Omega(n)-\omega(n))^{2N+2}}{\omega^{N+1}(n)}\right)\right\}$$

for any fixed integer $N \ge 1$ and suitable constants $d_{r,k}$, which may be explicitly evaluated. Now we substitute (15) in S_1, and similarly as in the proof of (14) we use (12) to show that the summation condition $\Omega(n)-\omega(n) \le \sqrt{\omega(n)}$ after this substitution may be omitted. Hence we shall have (with $N = M$)

$$(16)\qquad F(x) = O\left(\frac{x\log x}{(\log\log x)^{M+1}}\right) +$$

$$+ \sum_{2\le n\le x} 2^{\omega(n)} e^{\frac{1}{2}(\Omega(n)-\omega(n))} \left\{1 + \sum_{k=1}^{M}\sum_{r=k+1}^{2k} d_{r,k} \frac{(\Omega(n)-\omega(n))^r}{\omega^k(n)}\right\}.$$

Here the terms corresponding to the sums over k and r are directly evaluated by applying the Lemma with $c = 2,\ d = e^{\frac{1}{2}}$, and they will contribute the sum over j on the right-hand side of (7). There remains yet in (16) the sum of $f(n) := 2^{\omega(n)} e^{\frac{1}{2}(\Omega(n)-\omega(n))}$. It can be evaluated without difficulty directly, when one notes that, for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p \left(1 + 2\sum_{j=1}^{\infty} e^{\frac{1}{2}(j-1)} p^{-js}\right) = \zeta^2(s) H(s),$$

where

$$\begin{aligned} H(s) &= \sum_{n=1}^{\infty} h(n) n^{-s} = \prod_p \left(1 - p^{-s}\right)^2 \left(1 + 2\sum_{j=1}^{\infty} e^{\frac{1}{2}(j-1)} p^{-js}\right) \\ &= \prod_p \left(1 - p^{-s}\right)^2 \left(1 + \frac{2}{p^s - e^{\frac{1}{2}}}\right) \end{aligned}$$

is a Dirichlet series which is absolutely convergent for $\operatorname{Re} s > \frac{1}{\log 4}$.

Since $\zeta^2(s) = \sum_{n=1}^{\infty} d(n) n^{-s}$ $(\operatorname{Re} s > 1)$ and (4) holds, it follows that

$$\begin{aligned} \sum_{n \le x} f(n) &= \sum_{n \le x} \sum_{\delta \mid n} d(\delta) h(\frac{n}{\delta}) = \sum_{n \le x} h(n) \sum_{m \le \frac{x}{n}} d(m) \\ &= \sum_{n \le x} h(n) \left(\frac{x}{n} \log \frac{x}{n} + (2\gamma - 1)\frac{x}{n} + O\left((\frac{x}{n})^{\frac{1}{2}}\right)\right) \\ &= H(1) x \log x + O(x), \end{aligned} \tag{17}$$

which is sufficiently sharp for our purposes. Therefore if we insert (17) into (16) we obtain the assertion of the Theorem.

REFERENCES

1. J.-M. De Koninck, A. Ivić: *Topics in arithmetical functions.* Mathematics Studies 43, North-Holland, Amsterdam, 1980.
2. A. Ivić: *Sums of products of arithmetical functions.* Publ. Inst. Math. (Belgrade) **44** (**55**), 1987, 31-41.
3. A. Selberg: *Note on a paper by L.G. Sathe.* J. Indian Math. Soc., **18** (1954), 83-87.

Department of Mathematics,
Faculty of Mining and Geology,
University of Belgrade,
Đušina 7, 11000 Belgrade,
Yugoslavia

(Received November 11, 1992)