Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 4 (1993), 28–34.

A NOTE ON JENSEN'S DISCRETE INEQUALITY

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A refinement of Jensen's discrete inequality and some natural applications are given.

1. INTRODUCTION

The main aim of this paper is to point out a refinement of the famous Jensen's inequality which says that:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

where $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is convex on the interval I, $x_i \in I$ and $p_i \ge 0 (i = 1, \dots n)$ with $P_n > 0$. Some applications in connections with arithmetic mean–geometric mean inequality, with Ky Fan's well-known inequality and with Belmann–Bergström–Fan quasi-linear functionals are also established.

In a recent paper [**11**], the following refinement of (1) has been given:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{{P_n}^{k+1}} \sum_{i_1,\dots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right) \tag{2}$$

$$\le \frac{1}{{P_n}^{k}} \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \le \cdots \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i),$$

where $f : C \subset X \mapsto \mathbf{R}$ is a convex mapping on a convex set C (C is a subset of a linear space X) $p_i \ge 0,\ x_i \in C\ (i = 1, \dots n)$ with $P_n := \sum_{i=1}^{n} p_i > 0$ and k is a positive integer such that $1 \le k \le n-1$.

[0]1991 Mathematics Subject Classification: 26D 15

Another improvement for weighted means was given in [**6**, Theorem 3] where it is shown that:

$$(3)\quad f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n{}^k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j}\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i),$$

for all $q_j \ge 0$ with $Q_k := \sum_{j=1}^{k} q_j > 0$.

For some interesting applications of these results we refer to [**6**–**7**] and [**11**] where further references are given.

2. MAIN RESULTS

We start start with the following result.

Theorem. *Let f, x_i, p_i be as above and let α_i, β_i be nonnegative real numbers with $\alpha_i + \beta_i > 0$ for all i, $j = 1, \dots, n$. Then we have the following inequalities*:

$$(4)\qquad f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{P_n{}^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right)$$

$$\le \frac{1}{P_n{}^2}\sum_{i,j=1}^{n} p_i p_j \frac{1}{2}\left(f\left(\frac{\alpha_i x_i + \beta_j x_j}{\alpha_i + \beta_j}\right) + f\left(\frac{\beta_j x_i + \alpha_i x_j}{\alpha_i + \beta_j}\right)\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).$$

Proof. The Jensen inequality for double sums yields

$$f\left(\frac{1}{P_n{}^2}\sum_{i,j=1}^{n} p_i p_j \left(\frac{x_i + x_j}{2}\right)\right) \le \frac{1}{P_n{}^2}\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right).$$

Since

$$\frac{1}{P_n{}^2}\sum_{i,j=1}^{n} p_i p_j \left(\frac{x_i + x_j}{2}\right) = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i,$$

the first inequality in (4) is proven.

By the convexity of f on C we have:

$$\frac{1}{2}\left(f\left(\frac{\alpha_i x_i + \beta_j x_j}{\alpha_i + \beta_j}\right) + f\left(\frac{\beta_j x_i + \alpha_i x_j}{\alpha_i + \beta_j}\right)\right) \ge f\left(\frac{x_i + x_j}{2}\right)$$

for all $i, j = 1, \dots, n$. By multiplying this inequality with $p_i p_j \ge 0$ $(i, j = 1, \dots, n)$ and summing over i and j (from 1 to n), we derive the second inequality in (4).

To prove the last inequality in (4), we observe that:

$$f\left(\frac{\alpha_i x_i + \beta_j x_j}{\alpha_i + \beta_j}\right) \le \frac{\alpha_i f(x_i) + \beta_j f(x_j)}{\alpha_i + \beta_j}$$

and

$$f\left(\frac{\alpha_i x_j+\beta_j x_i}{\alpha_i+\beta_j}\right)\le\frac{\alpha_i f(x_j)+\beta_j f(x_i)}{\alpha_i+\beta_j}$$

for all $i,j=1,\dots,n$. By addition we get

$$\frac{1}{2}\left(f\left(\frac{\alpha_i x_i+\beta_j x_j}{\alpha_i+\beta_j}\right)+f\left(\frac{\beta_j x_i+\alpha_i x_j}{\alpha_i+\beta_j}\right)\right)\le\frac{f(x_i)+f(x_j)}{2}$$

for all $i,j=1,\dots,n$.

By multiplying this inequality with $p_i p_j\ge 0$ and summing over i and j (from 1 to n), we obtain the desired inequality.

Now, let consider the mapping $F:[0,1]\to\mathbf{R}$ given by

$$F(t):=\frac{1}{P_n{}^2}\sum_{i,j=1}^{n}p_i p_j f(tx_i+(1-t)x_j),$$

where $f:I\subseteq\mathbf{R}\to\mathbf{R}$ is as above, $x_i\in I$ and $p_i\ge 0$ $(i=1,\dots,n)$ with $P_n>0$. Then the following corollary holds.

Corollary. *Under above assumptions, for all* $t\in[0,1]$ *we have the inequality:*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right)\le\frac{1}{P_n{}^2}\sum_{i,j=1}^{n}p_i p_j f\left(\frac{x_i+x_j}{2}\right)\le F(t)\le\frac{1}{P_n}\sum_{i=1}^{n}p_i f(x_i).$$

The proof is obvious from the above theorem (choosing $\alpha_i=t$, $\beta_j=1-t$ $(i,j=1,\dots,n)$). We will omit the details.

Remark. It is easy to see, from the above corollary, that:

$$\sup_{t\in[0,1]}F(t)=F(0)=F(1)=\frac{1}{P_n}\sum_{i=1}^{n}p_i f(x_i)$$

and

$$\inf_{t\in[0,1]}F(t)=f\left(\frac{1}{2}\right)=\frac{1}{P_n{}^2}\sum_{i,j=1}^{n}p_i p_j f\left(\frac{x_i+x_j}{2}\right).$$

For other refinements of Jensen's inequality see the paper [**5**] where further references are given.

3. APPLICATIONS

I. 1. Let $x_i,\ p_i\ge 0$ $(i=1,\dots,n)$ with $P_n>0$. Then the following refinement of arithmetic mean–geometric mean inequality holds:

$$\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\ge\left(\prod_{i,j=1}^{n}\left(\frac{x_i+x_j}{2}\right)^{p_i p_j}\right)^{1/P_n{}^2}$$

$$\geq \left(\prod_{i,j=1}^{n} \left(\frac{(\alpha_i x_i + \beta_j x_j)^{1/2} (\beta_j x_i + \alpha_i x_j)^{1/2}}{\alpha_i + \beta_j} \right)^{p_i p_j} \right)^{1/{P_n}^2} \geq \left(\prod_{i=1}^{n} {x_i}^{p_i} \right)^{1/P_n},$$

for all α_i, $\beta_j \geq 0$ so that $\alpha_i + \beta_i > 0$ $(i, j = 1, \ldots, n)$. The equality holds in all inequalities if and only if $x_1 = x_2 = \cdots = x_n$.

2. Let $x_i \in \mathbf{R}$, $p_i \geq 0$ $(i = 1, \ldots, n)$ with $P_n > 0$ and $p \geq 1$. Then for all α_i and β_j $(i, j = 1, \ldots, n)$ as above, we have:

$$\sum_{i=1}^{n} p_i |x_i|^p \geq {P_n}^{p-2} \sum_{i,j=1}^{n} p_i p_j \left| \frac{x_i + x_j}{2} \right|^p$$

$$\geq {P_n}^{p-2} \sum_{i,j=1}^{n} p_i p_j \frac{1}{2} \left(\left| \frac{\alpha_i x_i + \beta_j x_j}{\alpha_i + \beta_j} \right|^p + \left| \frac{\beta_j x_i + \alpha_i x_j}{\alpha_i + \beta_j} \right|^p \right) \geq {P_n}^{p-1} \sum_{i=1}^{n} p_i |x_i|^p.$$

3. Let $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$. Then the following refinement of the well-known inequality due to Ky Fan [**3**] is valid:

$$\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} (1 - x_i) \geq \prod_{i,j=1}^{n} ((x_i + x_j)/(2 - x_i - x_j))^{1/n^2}$$

$$\geq \left(\prod_{i,j=1}^{n} \left(\frac{(\alpha_i x_i + \beta_j x_j)(\beta_j x_i + \alpha_i x_j)}{\big((1 - \alpha_i) x_i + (1 - \beta_j) x_j\big)\big((1 - \beta_j) x_i + (1 - \alpha_i) x_j\big)} \right)^{1/2} \right)^{1/n^2}$$

$$\geq \left(\prod_{i=1}^{n} x_i / \prod_{i=1}^{n} (1 - x_i) \right)^{1/n},$$

for all α_i, $\beta_j \geq 0$ so that $\alpha_i + \beta_j > 0$ $(i, j = 1, \ldots, n)$. The equality holds if and only if $x_1 = \cdots = x_n$.

4. In the recent paper [**1**], H. Alzer has established the following converse of Ky Fan's inequality:

$$\sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) \;\leq\; \prod_{i=1}^{n} (x_i/(1 - x_i))^{x_i / \sum_{i=1}^{n} x_k},$$

where $x_i \in (0, 1)$ $(i = 1, \ldots, n)$ and the equality holds in the above inequality if and only if $x_1 = \cdots = x_n$. We may improve this fact as in the sequel:

$$\sum_{i=1}^{n} x_i / \sum_{i=1}^{n} (1 - x_i) \;\leq\; \left(\prod_{i,j=1}^{n} \big((x_i + x_j)/(2 - x_i - x_j)\big)^{(x_i + x_j)/2} \right)^{1 / \left(n \sum_{k=1}^{n} x_k \right)}$$

$$\leq \prod_{i,j=1}^{n} \left(\left(\frac{\alpha_i x_i + \beta_j x_j}{(1-\alpha_i)x_i + (1-\beta_j)x_j} \right)^{\frac{\alpha_i x_i + \beta_j x_j}{2(\alpha_i+\beta_j)}} \right.$$

$$\left. \times \left(\frac{\alpha_i x_j + \beta_j x_i}{(1-\alpha_i)x_j + (1-\beta_j)x_i} \right)^{\frac{\alpha_i x_j + \beta_j x_i}{2(\alpha_i+\beta_j)}} \right)^{1/\left(n \sum_{k=1}^{n} x_k\right)}$$

$$\leq \prod_{i=1}^{n} \left(x_i/(1-x_i) \right)^{x_i/\left(\sum_{k=1}^{n} x_k\right)},$$

where α_i, β_j $(i,j = 1,\ldots,n)$ are as above.

The proofs of the above statements follow from (2) for the convex mapings: $f(x) := -\ln x$, $x > 0$; $f(x) := |x|^p$, $x \in \mathbf{R}$; $f(x) := -\ln(x/(1-x))$, $x \in (0, 1/2]$ and $f(x) := \ln(x/(1-x))^x$, $x \in (0,1)$.

II. Now, let X be a real linear space and K be a clin in X, i.e., a subset of X satisfying the conditions:

(K$_1$) $\quad x, y \in K$ imply $x + y \in K$;

(K$_2$) $\quad x \in K$, $\alpha \geq 0$ imply $\alpha x \in K$.

Let us also suppose that $\varphi : K \to \mathbf{R}$ is a quasi-linear functional on K, i.e. a mapping which satisfies the assumption:

$$\varphi(\alpha x + \beta y) \leq (\geq)\, \alpha\varphi(x) + \beta\varphi(y), \tag{5}$$

for all $x, y \in K$ and $\alpha, \beta \geq 0$.

We observe that such a functional is a convex (concave) maping on K but the converse implication is not true in general. We also observe that the following inequality holds:

$$\varphi\left(\sum_{i=1}^{n} p_i x_i\right) \leq (\geq) \sum_{i=1}^{n} p_i \varphi(x_i), \tag{6}$$

for all $p_i \geq 0$ and $x_i \in K$ $(i = 1,\ldots,n)$.

By the use of the above theorem, we can give the following improvement of (6):

Let φ be as above, $x_i \in K$, $p_i \geq 0$ $(i = 1,\ldots,n)$ and let α_i, β_i be non-negative real numbers with $\alpha_i + \beta_i > 0$ for all $i, j = 1,\ldots,n$. Then we have the inequalities:

$$\varphi\left(\sum_{i=1}^{n} p_i x_i\right) \leq (\geq) \frac{1}{2P_n} \sum_{i,j=1}^{n} p_i p_j \varphi(x_i + x_j) \tag{7}$$

$$\leq (\geq) \frac{1}{P_n} \sum_{i,j=1}^{n} \frac{p_i p_j}{\alpha_i + \beta_j} \cdot \frac{\varphi(\alpha_i x_i + \beta_j x_j) + \varphi(\beta_j x_i + \alpha_i x_j)}{2} \leq (\geq) \sum_{i=1}^{n} p_i \varphi(x_i).$$

As in [**13**], we shall use the following notations:

$\mathcal{M} = \{M \mid M$ is a positive definite matrix of order $n\}$,

$|M| =$ the determinant of the matrix M,

$|M|_k = \prod_{j=1}^{k} \lambda_j, \ k = 1, \ldots, n,$ where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of M with $\lambda_1 \le \cdots \le \lambda_n, \ |M|_n = |M|$;

$M(j) =$ the submatrix of M obtained by deleting the j^{th} row and j^{th} column of the matrix M;

$M[k] =$ the principal submatrix of M formed by taking the first k rows and columns of M; $M[n] = M, \ M[n-1] = M(n), \ M[0] =$ the identity matrix;

BBF = the class of Bellman–Besgström–Fan quasi-linear functionals σ_i, δ_j and ν_k defined on M by:

$$\begin{aligned} \sigma_i(M) &:= |M|_i^{1/i}, && i = 1, \ldots, n; \\ \delta_j(M) &:= |M|/|M(j)|, && j = 1, \ldots, n: \\ \nu_k(M) &:= (|M|/|M[k]|)^{1/(n-k)}, && k = 1, \ldots, n, \end{aligned}$$

respectively.

It is evident that M is closed under addition and multiplication by a positive number, i.e. M is a clin. Now, quasi-linearity of BBF–functionals follows from results given in [**13**]:

$$\varphi(pM_1 + qM_2) \ge p\varphi(M_1) + q\varphi(M_2),$$

for all $M_1, M_2 \in \mathcal{M}, \ p, q \ge 0$ and $\varphi \in$ BBF (see also [**8**]).

In [**13**], C.L.Wang has obtained the following inequality:

$$\varphi\left(\sum_{i=1}^{m} p_i M_i\right) \ge \sum_{i=1}^{m} p_i \varphi(M_i) \ge P_m \prod_{i=1}^{m} \Big(\varphi(M_i)\Big)^{p_i/P_m}$$

$p_i > 0 \ \ (i = 1, \ldots, m)$, which is an interpolation inequality for

$$\varphi\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i M_i\right) \ge \prod_{i=1}^{m} \Big(\varphi(M_i)\Big)^{p_i/P_m}. \tag{8}$$

Note that (8) is also a generalization of a rezult from [**9**].

By the use of inequality (7), we can improve (8) as follows:

$$\begin{aligned} \sum_{i=1}^{m} p_i \varphi(M_i) \ &\ge\ \frac{1}{P_m} \sum_{i,j=1}^{m} \frac{p_i p_j}{\alpha_i + \beta_j} \cdot \frac{\varphi(\alpha_i M_i + \beta_j M_j) + \varphi(\beta_j M_i + \alpha_i M_j)}{2} \\ &\ge\ \frac{1}{2P_m} \sum_{i,j=1}^{m} p_i p_j \varphi(M_i + M_j) \ge \varphi\left(\sum_{i=1}^{m} p_i M_i\right), \end{aligned}$$

where $M_i \in \mathcal{M},\ p_i \geq 0,\ \alpha_i + \beta_i > 0\ \ (i, j = 1, \ldots, m)$ and $\varphi \in \mathrm{BBF}$.

REFERENCES

1. H. Alzer: *A converse of Ky Fan's inequalities.* C.R. Math. Rep. Acad. Sci. Canada, **9** (1989), 1–3.
2. R. E. Barlow, A. W. Marschall, F. Proschan: *Some inequalities for starshaped and convex functions.* Pacific J. Math., **29** (1969), 14–42.
3. E. F. Beckenbach, R. Bellman: *Inequalities,* 4-th ed., Springer Verlag, Berlin, 1983.
4. S. S. Dragomir: *A refinement of Jensen inequality.* G. M. Metod. (Bucharest), **10** (1989), 190–191.
5. S. S. Dragomir, N. M. Ionescu: *On some inequalities for convex dominated functions.* Anal. Num. Theor. Approx., **19** (1990), 21–28.
6. S. S. Dragomir: *An improvement of Jensen's inequality.* Bull. Math. Soc. Sci. Math. Roumanie, **34** (82) (1990), 291–296.
7. S.S.Dragomir: *Some refinements of Ky Fan's inequality.* J. Math. Anal. Appl., **163** (1992), 317–321.
8. D. S. Mitrinović, J. E. Pečarić: *Determinental inequalities of Jensen's type.* An. Öster. Akad. Wiss. Math. Klasse **125** (1988), 75–78.
9. L. Mirsky: *An inequality for positive definite matrices.* Amer. Math. Montly, **62** (1955), 428–430.
10. J. E. Pečarić: *Inverse of Jensen–Steffensen's inequality.* Glas. Mat., **1**. (1981), 229–233.
11. J. E. Pečarić, S. S. Dragomir: *A refinement of Jensen inequality and applications.* Studia Math. Univ. „Babeş–Bolyai", **34** (1) (1989), 15–19.
12. C. L. Wang: *On a Ky Fan inequality for the complementary A.–G. type and its variants.* J. Math. Anal. Appl., **73** (1980), 501–505.
13. C. L. Wang: *Extension of determinantal inequalities.* Utilitas Math., **13**, (1978), 201–210.

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(Received Septembar 15, 1992)

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