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ON A ZIRAKZADEH INEQUALITY RELATED TO TWO TRIANGLES INSCRIBED ONE IN THE OTHER

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We note that a conjecture from [1] (inequality XII. 1.20, p. 347) was proved in 1960. by A. Zirakzadeh. Stronger inequalities are obtained in this paper.

A conjecture due to Mao–Qi Tao and Jing–Zhong Zhang [**1**, p. 347] is stated as follows:

Let three points P, Q, R be on the sides BC, CA, AB of a triangle ABC, respectively. If $QA + AR = RB + BP = PC + CQ$, then

$$QR + RP + PQ \geq \frac{1}{2}(a + b + c), \tag{1}$$

with equality if and only if the triangle is equilateral and P, Q, R are the midpoints of the sides of ABC.

However, the inequality (1) was first proved in 1960. by A. Zirakzadeh [**2**], and his proof is very difficult. A simpler proof was given by Zhen–Bing Zeng [**3**].

Now, we shall prove the following theorem:

Theorem. *We have*

$$QR + RP + PQ \geq \frac{a+b+c}{3}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right); \tag{2}$$

$$(QR + RP + PQ)^3 \geq \frac{9}{8}(a^3 + b^3 + c^3), \tag{3}$$

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with equality if and only if the triangle is equilateral and P, Q, R are the midpoints of the sides of ABC.

Proof. Let MN be the projection of QR on the side BC. Then

$$QR \geq MN = a - (BR \cdot \cos B + CQ \cdot \cos C).$$

In the same way, we have

$$RP \geq b - (CP \cdot \cos C + AP \cdot \cos A), \quad PQ \geq c - (AQ \cdot \cos A + BP \cdot \cos B),$$

and therefore

$$QR + RP + PQ \geq \frac{1}{3}(a+b+c)(3 - \cos A - \cos B - \cos C). \tag{4}$$

It is known that $\cos A + \cos B + \cos C = \frac{R+r}{R}$ (see [**1**, p. 55]). Therefore, (4) is equivalent to $QR + RP + PQ \geq \frac{2s(2R-r)}{3R}$.

On the other hand

$$\begin{aligned} &\frac{a+b+c}{3}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \\ = &\frac{(a+b+c)\Big(a(a+b)(a+c) + b(b+c)(b+a) + c(c+a)(c+b)\Big)}{3(b+c)(c+a)(a+b)} \\ = &\frac{(a+b+c)(a^3+b^3+c^3+a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+3abc)}{3(b+c)(c+a)(a+b)}, \end{aligned}$$

and it is know that

$$a^3 + b^3 + c^3 = 2s(s^2 - r^2 - 6Rr)$$

and

$$(b+c)(c+a)(a+b) = 2s(s^2 + r^2 + 2Rr)$$

(see [**1**], pp. 52-53). Therefore, (2) and (3) are equivalent to

$$QR + RP + PQ \geq \frac{4s(s^2 - r^2 - Rr)}{3(s^2 + r^2 + 2Rr)}, \tag{2'}$$

$$(QR + RP + PQ)^3 \geq \frac{9}{4}s(s^2 - 3r^2 - 6Rr), \tag{3'}$$

respectively. We only need to prove the following two inequalities:

$$\frac{2s(2R-r)}{3R} \geq \frac{4s(s^2 - r^2 - Rr}{3(s^2 + r^2 + 2Rr)}, \tag{5}$$

$$\left(\frac{2s(2R-r)}{3r}\right)^3 \geq \frac{9}{4}s(s^2 - 3r^2 - 6Rr). \tag{6}$$

The inequality (5) is equivalent to $(2R-r)(s^2+r^2+2Rr) \geq 2R(s^2-r^2-Rr)$, i.e.

$$r(6R^2+2Rr-r^2-s^2) \geq 0. \tag{7}$$

Using the Gerretsen inequality [**1**, p. 45]

$$s^2 \leq 4R^2+4Rr+3r^2 \tag{8}$$

and Chapple–Euler inequality $R \geq 2r$, then (7) follows, and we have proved the inequality (2).

The inequality (6) is equivalent to

$$\begin{aligned} H(s^2) &\equiv 32s^2(2R-r)^3 - 243R^3(s^2-3r^2-6Rr) \\ &= 32(2R-r)^3 - 243R^3s^2 + 243R^3(3r^2+6Rr) \geq 0. \end{aligned} \tag{9}$$

If $32(2R-r)^3 \geq 243R^3$, then (9) is obvious; if $32(2R-r)^3 < 243R^3$, then by the Gerretsen inequality (8), we only need to prove $H(4R^2+4Rr+3r^2) \geq 0$. Now

$$\begin{aligned} H(4R+4Rr+3r^2) &= 32(4R^2+4Rr+3r^2)(2R-r)^3 - 243R^3(4R^2-2Rr) \\ &= 2(2R-r)(R-2r)(13R^3+26R^2r+52Rr^2-24r^3) \geq 0, \end{aligned}$$

and (9) follows. Hence (3) is verified.

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