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CHARACTERIZATION OF LINEAR INVOLUTIONS

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In this note we give a brief account of linear involutions.

Let A be a linear transformation on an n-dimensional unitary space such that $A^p = I$, where p is the least positive integer, and I is the identity. Then A is called a linear involution of order p. Properties of linear involutions have been studied and can be found scattered in the literature. In this note we give a brief account of them.

Definitions and Notation. An n-dimensional unitary space will be denoted by E_n. Greek and Latin letters will denote vectors and scalars respectively. Linear transformations will be indicated by capital letters. Let $\mathcal{B}$ be a basis for E_n. Then the matrix of a linear transformation A with respect to $\mathcal{B}$ will be $[A]_{\mathcal{B}}$. In what follows all transformations are linear. The inner product of ξ and η will be denoted by $\langle \xi, \eta \rangle$. The adjoint A^* of a linear transformation A is defined by $\langle A\xi, \eta \rangle = \langle \xi, A^*\eta \rangle$. The direct sum of two subspaces S and T will be denoted by $S \oplus T$. Other definitions will be given whenever needed.

2. Proper Values of a Linear Involution. *Let A be a linear involution of order p on E_n. Then every proper value od A is a p-th root of unity.*

Proof. Let ξ be a proper vector of A corresponding to proper value a. Then $A\xi = a\xi$, $\xi \neq 0$, which implies $A^p\xi = a^p\xi = \xi, \quad \xi \neq 0$, or $(a^p - 1)\xi = 0, \quad \xi \neq 0$. Thus $a^p - 1 = 0$, which proves the proposition.

3. Theorem. *Let $A \neq I$ be a Hermitian transformation on E_n such that $A^k = I$, $k \geq 2$. Then A is an involution of order* 2.

Since the proper values of A are real, they must be ± 1. (Note that A is a symmetry with respect to a proper subspace of E_n.)

4. Theorem. *Let A be a positive transformation on E_n such that $A^p = I$. Then $A = I$.*

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Since the proper values of A are positive and also roots of unity, they all must be 1. Thus $A = I$.

5. Theorem. *Let A be a normal transformation and an involution of order p on E_n. Then A is unitary (an isometry).*

Proof. Since $A^*A = AA^*$ and $A^p = I$, and $(A^*A)^p = I$. It is clear that A^*A is positive, and thus by Theorem 4 we must have $A^*A = I$, which proves the theorem.

6. Theorem. *A necessary and sufficient condition for a linear transformation A on E_n to be an involution of order p is that $A = a_1P_1 + \cdots + a_kP_k$, where a_i, $i = 1, \ldots, k$ is a p-th root of unity, and $P_1, \ldots, P_k$ are projections such that $P_iP_j = 0$, $i \neq j$ and $P_1 + \cdots + P_k = I$.*

Proof. The sufficiency is obvious.

Now let $A^p = I$, and $a_1, \ldots, a_k$ be distinct proper values of A with algebraic multiplicies $m_1, \ldots, m_k$ respectively. Then there exist subspaces $S_1, \ldots, S_k$ such that $E_n = S_1 \oplus \cdots \oplus S_k$, each S_i is invariant under A, and $A = a_iI + N_i$ on S_i [**1**]. Essentially, we are using the JORDAN canonical form of A. To prove the necessity, we must show that $N_i = 0$, $i = 1, \ldots, k$.

Suppose for some i the index of nilpotency of N_i is $m \geq 2$. Then there is an $\eta \in S_i$, $\eta \neq 0$ such that ${N_i}^m\eta = 0$, and ${N_i}^{m-1}\eta \neq 0$. Let ${N_i}^{m-1}\eta = \eta_1 \neq 0$ and ${N_i}^{m-2}\eta = \eta_2 \neq 0$. Then $A\eta_1 = a_i\eta_1$, and $A\eta_2 = \eta + a_1\eta_2$. This implies that $A^k\eta_1 = {a_i}^k\eta_1$, $A^k\eta_2 = k{a_i}^{k-1}\eta_1 + {a_i}^k\eta_2$, where k is a positive integer. In particular $a^p\eta_2 = p{a_i}^{p-1}\eta_1 + {a_i}^p\eta_2$. Since $A^p = I$ and ${a_i}^{p-1} = 1$, the above equality implies ${a_i}^{p-1}\eta_1 = 0$. This contradicts the fact that $a_i \neq 0$ and $\eta_1 \neq 0$. Therefore $N_i = 0$.

Let P_i be the projection on S_i along $S_1 \oplus \cdots \oplus S_{i-1} \oplus S_{i+1} \oplus \cdots \oplus S_k$, $i = 1, \ldots, k$. Then $A = a_1P_1 + \cdots + a_kP_k$.

7. Theorem. *Let $[A]$ be the matrix of the linear transformation A on E_n with respect to basis $\mathcal{B}$. Then A is a linear involution of order n whose proper values are distinct n-th roots of unity if and only if all the principal k-rowed minors of $[A]$, $k = 1, \ldots, n-1$ are zero and* $\det A = (-1)^{n-1}$. (By [**2**], p. 19, this means that the characteristic equation of A is $z^n - 1 = 0$.)

The proof is straightforward and will be omitted.

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