ON PERFECT PAIR STRUCTURES
IN THE SET OF SPANNING TREES
OF A GRAPH

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Structures in the set of all spanning trees of a graph, related to the concepts of perfect pairs, hybrid bases and superperfect pairs are considered. Variety of patterns and high regularity of these structures are underlined. An algorithm for generating basic data set for these structures is presented.

0. INTRODUCTION

Novak, L. A. and Gibbons, A. have recently introduced new concepts of perfect and superperfect pairs of spanning trees and hybrid bases in graphs ([**1**]–[**4**]). These concepts induce interesting structures in the set of all spanning trees of a graph, as it was pointed out in [**1**].

The main purpose of this paper is to underline surprising richness and high regularity of perfect pair structures, which seems to be an inherent property. Suitable condensed representations are used to illustrate global aspects of these structures.

An algorithm for generating necessary data is also presented. The input graphs are assumed to be non-oriented and 2-connected.

1. DEFINITIONS AND DENOTATIONS

We assume familiarity with the following basic notions of graph theory:

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circuit, cutset, spanning tree, fundamental circuit and *fundamental cutset.* All these notions are treated as subsets of the edge set of a graph. Let A be a subset of the edge-set E of a graph G. Then *rank(A)* denotes the non-negative integer equal to the largest circuitless subset of A. The *complement* of A, denoted by A^*, is the set-difference $E\backslash A$. Cardinality of the subset A is denoted by $|A|$. The edge-sets will be denoted without brackets and commas.

Let t be a spanning tree of a graph G. *Diameter* of the spanning tree t is equal to $rank(t^*)$.

A pair of spanning trees (t_1, t_2) is said to be a *perfect pair* ([1]) if none of the fundamental circuits with respect to t_1 and t_2, defined by an edge in $(t_1^* \cap t_2^*)$, contains an edge in $(t_1 \cap t_2)$.

A pair of spanning trees (t_1, t_2) is said to be a *superperfect pair* ([2]) iff any fundamental cutset that edges in $(t_1 \cap t_2)$ form with edges in either t_1^* or t_2^* and any circuit that edges in $(t_1^* \cap t_2^*)$ form with edges in either t_1 or t_2 − are disjoint.

A subset b of edges of a graph G is said to be a *hybrid base* [3] (a maximal doubly independent set) of G if it is maximal circuitless and cutsetless subset of edges in the sense that each superset of b contains circuit and/or cutset.

The concept of perfect pair of trees is a natural generalization of so-called maximally distant pair of trees. The concept of superperfect pair of trees provides a refinement of a perfect pair of trees towards the concept of maximally distant pair of trees. This notion is also closely related to the notion of maximal double independence in graph.

The following two assertions give alternative definitions of perfect pairs and superperfect pairs:

Assertion 1. [1] A pair of spanning trees (t_1, t_2) is a perfect pair iff
$$\mathrm{rank}(t_1^*) \;=\; |t_1\backslash t_2| \;=\; |t_2\backslash t_1| \;=\; \mathrm{rank}(t_2^*).$$

Assertion 2. [2] A pair of spanning trees is a superperfect pair (*SP-pair*) iff the both set-differences $t_1\backslash t_2$ and $t_2\backslash t_1$ are hybrid bases.

A perfect pair (t_1, t_2) is called *ordinary perfect* (*OP-pair*), respectively *half-superperfect* (*HSP-pair*), if none, respectively only one, of the set differences $t_1 \backslash t_2$ and $t_2 \backslash t_1$ − is a hybrid base.

Note that any superperfect pair is a perfect pair and that any hybrid base is a set-difference of spanning trees within a perfect pair. The converse statements are not generally true.

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Perfect pairs induce an incidence relation in the set of all spanning trees of the input graph and hence form a tree graph. Hybrid bases induce an edge orientation

of that graph.

Let $P(G, d)$ denote a tree graph defined in the following way:

— the vertices are associated to the spanning trees of diameter d of the graph G.

— two vertices of $P(G, d)$ are joined by an edge iff their corresponding trees constitute a perfect pair of G.

— the edge associated to a perfect pair (t_1, t_2) (where t_1 and t_2 are some two spanning trees of G, both of diameter d), is oriented from t_1 to t_2 iff $t_1 \setminus t_2$ is a hybrid base of G. If both $t_1 \setminus t_2$ and $t_2 \setminus t_1$ are hybrid bases, then this edge is doubly oriented.

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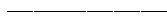
As an introductory example for the notions introduced above, we consider the graph G_1 in Fig. 1.
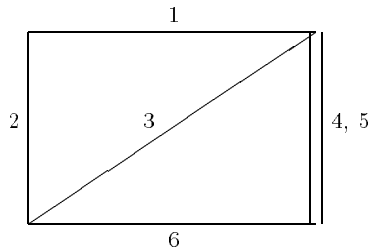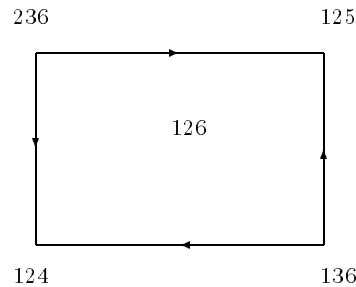

Figure 1. G_1

Figure 2. $P(G_1, 2)$

The spanning trees 134, 135, 146, 156, 234, 235, 246 and 256 are cutsetless and have diameter 3. Since they are at the same time maximal circuitless sets of edges, they are all hybrid bases. The complementary pairs of spanning trees among these eight — are superperfect.

The set 36 is a special hybrid base. Note that the addition of the edges 1 or 2 to the set 36 would give a cutset, while the addition of the edges 4 or 5 would give a cycle.

The spanning trees 124, 125, 136, 236 and 126 have diameter 2. The set-differences of 136 and 236 with any of 124 and 125 are equal to the special hybrid base. There are no other perfect pairs and hybrid bases. The vertex 126 of the graph $P(G_1, 2)$ is isolated (Fig. 2.).

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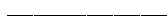
The graphs $P(G, d)$, together with the additional data which could be derived from them (vertex degrees, connected components, cycles, etc.) are considered merely as a *basic data set* for generating perfect pair structures. Under "perfect pair structures" we mean "nice" representations of these graphs, with as much regularities pointed out as possible. We propose the precise definition of "beauty of a structure" to be a topic for future work.

2. ALGORITHM

An algorithm for generating the basic data set of perfect pair structures of a 2-connected graph G is presented. The set consists of graphs $P(G, d)$ for all possible diameters d, as well as of some additional data obtained by analyzing these graphs.

The algorithm primarily finds the perfect pairs of spanning trees and partitions them into subclasses of OP-, HSP- and SP-pairs. This part of the algorithm is based on Assertions 1. and 2. and two procedures, which are used for

— enumeration of all trees, combined with sorting them into classes with respect to their diameters   and for

— testing whether a set of edges of a graph is a hybrid base .

The space-optimal algorithms developed in [5] and [6] respectively may be used for these purposes.

Let $D(G)$ denote the set of all diameters of spanning trees, associated to a 2-connected graph G . According to the property pointed out by HARARY, MOKKEN and PLANHOLT [5], $D(G)$ is *dense* in the sense that for any pair of diameters $k, j \in D(G)$, such that $k - j \geq 2$, there exists a diameter $s \in D(G)$, which is between k and j .

Suppose that a list of all the spanning trees of the input graph, sorted into classes by diameter, is given. Let k denote the diameter of a particular class. According to Assertion 1, a pair of trees (t_1, t_2) is a perfect pair of trees iff both t_1 and t_2 belong to the same diameter class, say class k, and the distance between t_1 and t_2 , i.e. $|t_1 \setminus t_2|$ — is equal to k. Since $|t_1| = |t_2|$, it is irrelevant which one of the set-differences $t_1 \setminus t_2$ and $t_2 \setminus t_1$ is in question.

We sketch the outlook of the algorithm:

```
(*  Generate   Catalogue( G, d ) of  all spanning trees of G,
    which have diameter d, for each possible d; *)
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```
REPEAT
   Output a new  spanning tree t of G ; (* local algorithm ([6]) *)
   Determine diameter d(t) as the difference of number of vertices
   of G  and number_of_connected_components_of (G-t);
   Adjust  max_diameter and/or min_diameter if necessary;
   Put  t   in  Catalogue( G, d(t) )
UNTIL  all the spanning trees of G are exhausted ;
  (*  In  this way are obtained all the vertices of all the
      graphs P(G,d)  *)

FOR  d  :=   min_diameter  TO  max_diameter  DO  BEGIN
  (*  Construct the edges (including their orientation)
      of the graph P(G,d) by  listing all the perfect pairs
      of G  and determining  whether they are  OP-, HSP- or
      SP-pairs *)
  FOR  each  pair  {t1,t2} of  trees in Catalogue( G, d )  DO
    IF   |t1 - t2|  =   diameter  THEN  BEGIN
      Check whether t1-t2 is a hybrid base;
      Check whether t2-t1 is a hybrid base;     (* [7] *)
      Output {t1,t2} as an OP-pair,
          a HSP-pair (with known orientation), or a SP-pair
    END;

  (*  Analyse the graph P(G,d)  *)
    Separate its connected components;
    FOR each vertex determine the number of its incident
        edges in the four classes:  in-, out-, doubly
        oriented and non-oriented;
    Within each connected component
        sort its vertices w.r.t. the quadruples of
        numbers determined in the previous step;
    Label the vertices of each connected component
        by 1,2,... in accordance with the obtained order;
    Using the labelled neighbours of each vertex, determine
        the isomorphisms between the connected components,
        in the cases when they exist;
    Generate the picture of a connected  component in each
        isopmorphic class;
END;  (* FOR d *)
```

Instead of finding an arbitrary picture, we would prefer in the last step to find a "nice" representation, as symmetric as possible. An extension of algorithm in that direction would lead to an automated generation of the structure itself.

3. AN ILLUSTRATIVE EXAMPLE

The perfect pair structure corresponding to the graph G_2 (Figure 3.) is described in this section. The basic data set for determining the structure was generated by applying the above algorithm (implemented in PASCAL). The running time was less than a minute on a PC-486.
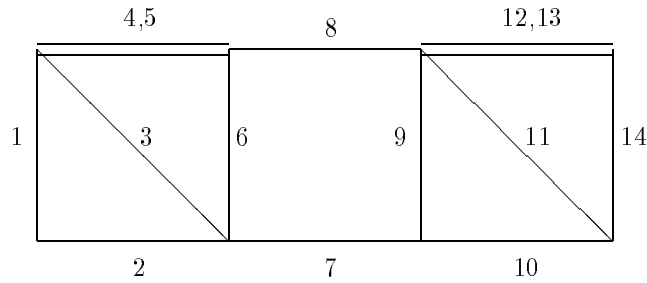

Figure 3. The graph G_2

The graph G_2 has spanning trees of four different diameters, 4 through 7.

The graph $P(G_2, 4)$ is a regular graph of degree 0, since the graph G_2 has not perfect pairs of diameter 4. The only five isolated vertices of $P(G_2, 4)$ are **1 2 7 8 9 10 12**, **1 2 7 8 9 10 13**, **1 2 7 8 9 10 14**, **1 2 7 8 9 11 14** and **1 2 7 8 10 11 14**.

The graph $P(G_2, 7)$ is a regular graph of degree 1 on 128 vertices. Each one of the 64 edges of this graph corresponds to a superperfect pair of complementary spanning trees of diameter 7.

The graphs $P(G_2, 5)$ and $P(G_2, 6)$ are considered in more detail. The graph $P(G_2, 5)$ includes both OP- and HSP-pairs, while the graph $P(G_2, 6)$ includes both HSP- and SP-pairs. The main data for these two graphs are given in the following table:

	NT	IT	OP	HSP	SP
$P(G_2, 5)$	128	40	160	168	—
$P(G_2, 6)$	272	40	—	320	32

The columns of the table respectively contain the number NT of spanning trees, the number IT of "isolated" trees (apart from perfect pairs), the number of OP-pairs, the number of HSP-pairs and the number of SP-pairs.

4.1. Diameter 6

Apart from 40 isolated vertices, the graph $P(G_2,6)$ has sixteen connected components. Eight of them (with 12 vertices each) are all isomorphic to the first one, while the other eight components (with 17 vertices each) are all isomophic to the second one ("snow-flake") of the components shown in Fig. 4. The "snow-flakes" include both HSP- and SP-pairs.
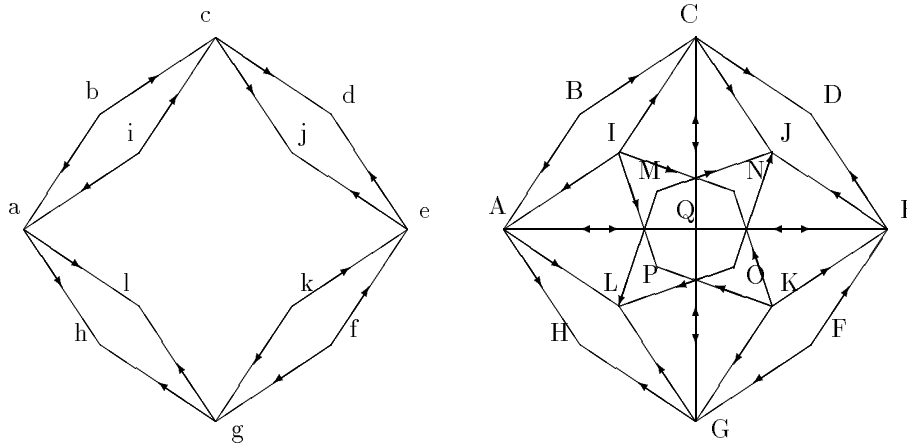

Figure 4. Typical non-trivial components of $P(G_2,6)$

The vertices in Fig. 4 correspond to the trees given in the following list:

```
a = 2 3 6 8 9 11 12      e = 1 3 6 7 9 11 12      i = 1 2 4 7 10 13 14
b = 1 2 5 7 10 13 14     f = 1 2 5 8 10 13 14     j = 2 4 7 8 10 13 14
c = 1 3 6 8 9 11 12      g = 2 3 6 7 9 11 12      k = 1 2 4 8 10 13 14
d = 2 5 7 8 10 13 14     h = 1 5 7 8 10 13 14     l = 1 4 7 8 10 13 14

A = 1 3 4 7 9 10 13      G = 1 3 4 8 9 10 13      M = 1 3 4 8 9 10 14
B = 2 5 6 8 10 11 14     H = 2 5 6 7 8 12 14      N = 1 3 4 7 8 10 12
C = 1 3 4 7 9 10 12      I = 2 5 6 8 9 11 14      O = 1 3 4 7 9 10 14
D = 2 5 6 7 8 13 14      J = 2 5 6 7 8 11 13      P = 1 3 4 7 8 10 13
E = 1 3 4 8 9 10 12      K = 2 5 6 7 9 11 14      Q = 2 5 6 7 8 11 14
F = 2 5 6 7 10 11 14     L = 2 5 6 7 8 11 12
```

The remaining seven 12-vertex components are obtained from the given one by mapping the ordered triple of edges (10,13,14) to (9,11,12), (9,13,14), (10,11,12), (10,11,13), (9,12,14), (9,11,13) and (10,12,14) respectively.

The remaining seven 17-vertex components are obtained from the given one by mapping the ordered triple of edges (2,5,6) to (2,3,5), (2,4,6), (2,3,4), (1,3,4), (1,4,6), (1,3,5) and (1,5,6) respectively.

It is interesting to note that the last eight triples are coincident with the spanning trees of diameter 3 of the graph G_1, while the above eight triples correspond to the spanning trees of diameter 3 of a subgraph, which is a copy of G_1.

It might be also interesting to compare these observations with the analogous observations for the graph G_3, which is obtained from the graph G_2 by moving the edge 11 to the position of "the other diagonal of the right square". The graph G_3 possesses a vertical axis of symmetry.

The graph $P(G_3, 6)$ has sixteen non-trivial components with 12 vertices each, which are all isomorphic to the smaller non-trivial component of G_2. The first eight of these components can be obtained from the corresponding 12-vertex components og G_2 simply by replacing the edges 10 and 11, while the other eight components can be obtained from the first eight by applying the above observed symmetry.

4.2. Diameter 5

The graph $P(G_2, 5)$ is by far the most complex.

This graph has 40 isolated vertices and two large connected components, with 46 and 42 vertices respectively, both of which have both oriented and non-oriented edges. These two components are denoted by I and II and their global scheme is given in Figure 5. The single vertices are denoted by   (the same denotation is applied in figures 6. − 9.)
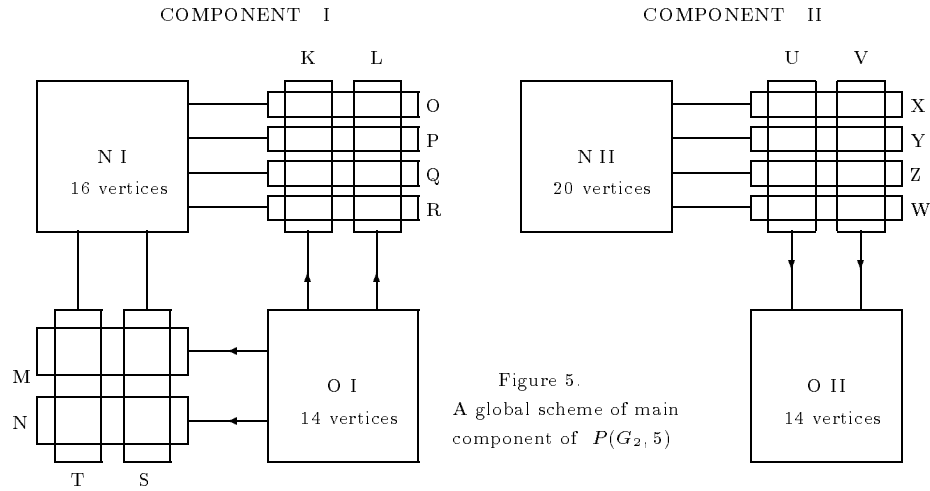

Figure 5.
A global scheme of main
component of $P(G_2, 5)$

N I = the purely non-oriented part of Component I (similarly N II)
O I = the purely   oriented part of Component I (similarly O II)

The number (v, e) of vertices and edges in the two components, partitioned

w.r.t. to the orientation of the edges − is given in the following table:

	oriented part	non-oriented part
Component I	(30, 104)	(32, 72)
Component II	(22, 64)	(28, 88)

For example, the number 104 in the table denotes the number of oriented edges in Component I. Note that 16 vertices of Component I, as well as 8 vertices of Component II, which are incident both tho oriented and non-oriented edges − are counted twice ($46 = 30 + 32 − 16$ and $42 = 22 + 28 − 8$).

These four parts are described by five figures (Fig. 6. - 10.). We explain in more detail the interconnections between them.

For the sake of clearness, the edges of oriented part of Component I are shown by two figures (6. and 7.); each one of the vertices A,B,...,F in Fig. 6. denotes the same spanning tree as the corresponding vertex in Fig. 7.

The denotation $\boxed{\begin{array}{c} n \\ o,i,u \end{array}}$ stands for a collection of n vertices, all of which have out-degree o , in-degree i and unoriented degree u. This denotation is used as a generalization of vertex. All the vertices within a box have the same relationship with the neighbouring vertices. Thus each (oriented or non-oriented) edge attached to an $n−$vertex box replaces n such edges attached to the vertices (spanning trees) within that box.

The following 24 spanning trees of diameter 5 of G_2 are incident both to oriented and non-oriented edges: (sixteen of them within Component I and the last eight within Component II).

```
a1  =  1 4 7 8 9 10 12        a5  =  2 4 7 8 9 10 12
a2  =  1 4 7 8 9 10 13        a6  =  2 4 7 8 9 10 13
a3  =  1 5 7 8 9 10 12        a7  =  2 5 7 8 9 10 12
a4  =  1 5 7 8 9 10 13        a8  =  2 5 7 8 9 10 13
b1  =  1 2 4 7 9 10 12        b5  =  1 2 4 8 9 10 12
b2  =  1 2 4 7 9 10 13        b6  =  1 2 4 8 9 10 13
b3  =  1 2 5 7 9 10 12        b7  =  1 2 5 8 9 10 12
b4  =  1 2 5 7 9 10 13        b8  =  1 2 5 8 9 10 13
c1  =  1 3 6 7 9 10 12        c5  =  1 3 6 7 9 10 13
c2  =  1 3 6 8 9 10 12        c6  =  1 3 6 8 9 10 13
c3  =  2 3 6 7 9 10 12        c7  =  2 3 6 7 9 10 13
c4  =  2 3 6 8 9 10 12        c8  =  2 3 6 8 9 10 13
```

The following eight 4-sets and eight 2-sets of spanning trees intersect as shown in Fig. 5. The boxes denoted to these sets are also shown in Figs. 6.-10.

```
L = a1 a2 a3 a4          O = a1 a3      P = a2 a4
K = a5 a6 a7 a8          Q = a6 a8      R = a5 a7

M = b1 b2 b3 b4          N = b5 b6 b7 b8
T = b1 b3 b5 b7          S = b2 b4 b6 b8

U = c1 c2 c3 c4          V = c5 c6 c7 c8
Y = c1 c5    Z = c2 c6   X = c3 c7       W = c4 c8
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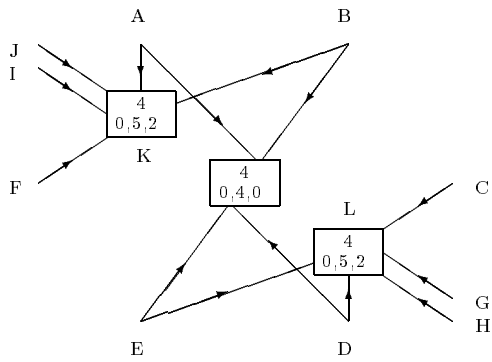

Figure 6. Oriented part of Component I
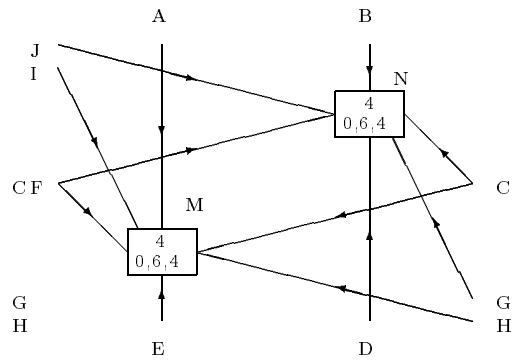of $P(G_2, 5)$ (first "level")

Figure 7. Oriented part of Component I
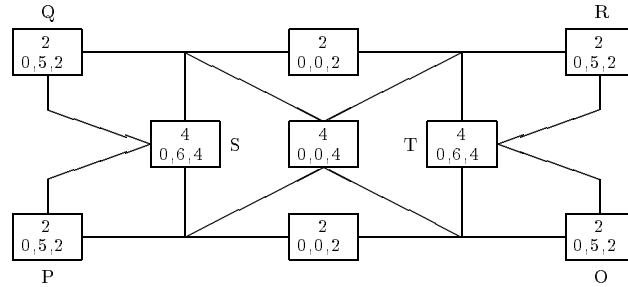of $P(G_2, 5)$ (second "level")

Figure 8. Non-oriented part
of Component I of $P(G_2, 5)$

Figure 9. Oriented part of
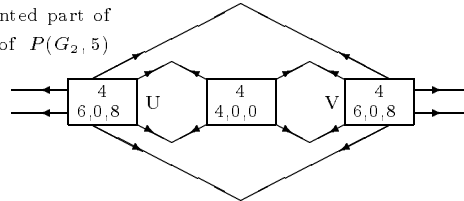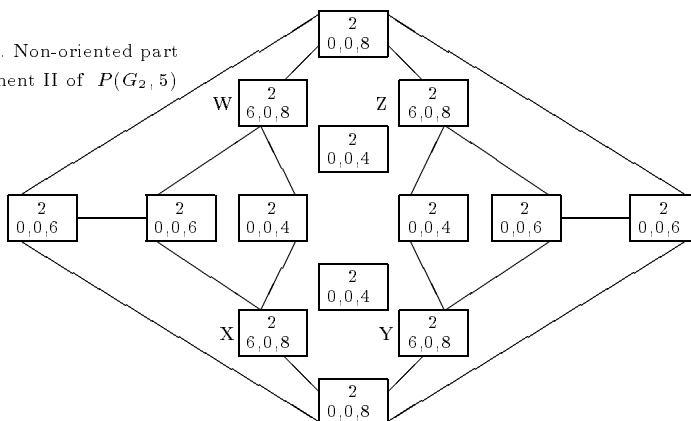Component II of $P(G_2, 5)$

Figure 10. Non-oriented part of Component II of $P(G_2, 5)$

4.3. Observations on symmetry

All the considered connected components of the perfect pair graphs are centrally symmetric. Let $[x, y]$ denote the transposition of elements x and y.

The permutation $[1, 2] * [4, 5]$ induces a central symmetry of $P(G_1, 2)$, although $[4, 5]$ is the only non-trivial edge-automorphism of G_1.

The components of $P(G_2, 6)$ shown in Figure 4. have the central symmetry $[7, 8] * [12, 13]$.

The four parts of $P(G_2, 5)$ on the Figures 6.−7., 8., 9. 10. have the central symmetries $[1, 2]*[7, 8]$, $[1, 2]*[12, 13]$, $[4, 5]*[12, 13]$ and $[1, 2]*[7, 8]$ respectively. In addition, the transpositions $[12, 14]$ and $[13, 14]$ map the quadruples S and T in Figure 8. to the central quadruple, while the transposition $[9, 10]$ maps the outer boxes of four "deltoids" in Figure 10. to their inner "deltoids".

On the other hand, note that the only non-trivial edge-automorphisms of G_2 are $[4, 5]$ and $[12, 13]$ (and their product).

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On the basis of these examples, we conjecture that the perfect pair graphs are generally more symmetric than the input graphs. More precisely, the perfect pair graphs have greater orbits of the automorphism groups. We think that it would be worth studying which new automorphic mappings (like transpositions $[1, 2]$ or $[7, 8]$ with G_2) arise when the collection of perfect pairs is considered instead of the underlying graph.

4. CONCLUSION

Structures in the set of trees of a graph induced by perfect pairs are considered and illustrated. The observed regularity motivates further investigations in the field. Development of algorithms for direct extraction of the global structure of tree graphs, combined with studying the relationships between the input graphs and their associated tree graphs — could be an important step towards examination of qualitative properties of the considered structures.

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