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ON SOME REFINEMENTS OF HADAMARD'S INEQUALITIES AND APPLICATIONS

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Some new refinements of Hadamard's inequalities and applications are given.

In the paper [**4**] S. S. DRAGOMIR introduced the following mapping associated to HADAMARD's inequalities:

$$H:\ [0,\,1]\to \mathbf{R},\ \ H(t):=\frac{1}{b-a}\int\limits_a^b f\left(tx+(1-t)\,\frac{a+b}{2}\right)\mathrm{d}x,$$

where $f\ :\ I\subseteq \mathbf{R}\to\mathbf{R}$ is a convex mapping on $I,\ a,\,b\in I$ with $a<b$ and he pointed out the following fundamental properties of this function. Namely, it was proved that:

(i) H is convex and monotonously increasing on $[0,\,1]$

and

$$\text{(ii)}\quad f\left(\frac{a+b}{2}\right)\le H(t)\le\frac{1}{b-a}\int\limits_a^b f(x)\,\mathrm{d}x,$$

for all $t\in[0,\,1]$.

In this paper we shall establish other facts connected to HADAMARD's result.

Theorem 1. *If f and H are as above, then*:

(i) *The following inequalities*:

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int\limits_{(3a+b)/4}^{(a+3b)/4} f(x)\,\mathrm{d}x\le\int\limits_0^1 H(t)\,\mathrm{d}t \tag{1}$$

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$$\leq \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)$$

hold;

(ii) *If f is differentiable on I then one has the inequalities*:

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - H(t) \leq (1-t)\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right) \tag{2}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - H(t) \leq \frac{(f'(b)-f'(a))(b-a)}{4} \tag{3}$$

for all $t \in [0,1]$.

Proof. (i) H being convex on $[0,1]$, Hadamard's inequalities yield

$$\frac{1}{b-a}\int_a^b f\left(\frac{2x+a+b}{4}\right)\mathrm{d}x \;=\; H\left(\frac{1}{2}\right) \leq \int_0^1 H(t)\,\mathrm{d}t \leq \frac{H(0)+H(1)}{2}$$
$$= \frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right),$$

which proves the statement.

(ii) Since f is convex on $[a,b]$, then

$$f\left(tx+(1-t)\frac{a+b}{2}\right) - f(x) \geq (1-t)\left(\frac{a+b}{2}-x\right)f'(x)$$

for all $t \in (0,1)$ and $x \in (a,b)$. Integrating on $[a,b]$, we derive

$$H(t) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \geq \frac{1-t}{b-a}\int_a^b \left(\frac{a+b}{2}-x\right)f'(x)\,\mathrm{d}x, \quad t \in [0,1].$$

Since a simple computation shows

$$\int_a^b \left(\frac{a+b}{2}-x\right)f'(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - (b-a)\frac{f(a)+f(b)}{2},$$

the inequality (2) is thus proven.

On the other hand, we have:

$$f\left(\frac{a+b}{2}\right) - f(a) \geq \frac{b-a}{2}f'(a) \quad \text{and} \quad f\left(\frac{a+b}{2}\right) - f(b) \geq \frac{a-b}{2}f'(b),$$

which gives, by addition:

$$f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\geq\frac{(f'(b)-f'(a))(a-b)}{4}.$$

Since $H(t)\geq f\left(\frac{a+b}{2}\right)$ for all $t\in[0,1]$, the above inequality yields that (3) is valid and the proof is finished.

Now, we shall introduce another mapping which is in connection with H and also with HADAMARD's result.

Let $f:\ I\subseteq\mathbf{R}\to\mathbf{R}$ be a convex function and $a,b\in I$ with $a<b$. Define the mapping:

$$G:\,[0,1]\to\mathbf{R},\ G(t):=\frac{1}{2}\left(f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left((1-t)\frac{a+b}{2}+tb\right)\right).$$

The following theorem contains some remarkable properties on this mapping.

Theorem 2. *Let f and G be as above. Then*:

(i) *G is convex and monotonously increasing on $[0,1]$;*

(ii) *We have*:

$$\inf_{t\in[0,1]}G(t)=G(0)=f\left(\frac{a+b}{2}\right)\tag{4}$$

and

$$\sup_{t\in[0,1]}G(t)=G(1)=\frac{f(a)+f(b)}{2};\tag{5}$$

(iii) *For all $t\in[0,1]$ the following inequality is valid*:

$$H(t)\leq G(t);\tag{6}$$

(iv) *One has the inequalities*:

$$\frac{2}{b-a}\int_{(3a+b)/4}^{(a+3b)/4}f(x)\,\mathrm{d}x\leq\frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\tag{7}$$

$$\leq\int_0^1G(t)\,\mathrm{d}t\leq\frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right);$$

(v) *If f is differentiable on I, then*:

$$0\leq H(t)-f\left(\frac{a+b}{2}\right)\leq G(t)-H(t)\qquad \textit{for all } t\in[0,1].\tag{8}$$

Proof. (i) Let $\alpha, \beta \geq 0$ with $\alpha+\beta=1$ and $t_1, t_2 \in [0,1]$. Therefore

$$\begin{aligned}
G(\alpha t_1+\beta t_2) &= \frac{1}{2}\Big(f\Big(\alpha\Big(t_1 a+(1-t_1)\frac{a+b}{2}\Big)+\beta\Big(t_2 a+(1-t_2)\frac{a+b}{2}\Big)\Big)\\
&+ f\Big(\alpha\Big((1-t_1)\frac{a+b}{2}+t_1 b\Big)+\beta\Big((1-t_2)\frac{a+b}{2}+t_2 b\Big)\Big)\\
&\leq \frac{1}{2}\Big(\alpha\Big(f\Big(t_1 a+(1-t_1)\frac{a+b}{2}\Big)+f\Big((1-t_1)\frac{a+b}{2}+t_1 b\Big)\Big)\\
&+ \beta\Big(f\Big(t_2 a+(1-t_2)\frac{a+b}{2}\Big)+f\Big((1-t_2)\frac{a+b}{2}+t_2 b\Big)\Big)\Big)\\
&= \alpha G(t_1)+\beta G(t_2),
\end{aligned}$$

which shows the convexity of G.

Now, since G is convex on $[0,1]$, then for all $t_1, t_2 \in (0,1)$ with $t_2 > t_1$ we have:

$$\begin{aligned}
&\Big(G(t_2)-G(t_1)\Big)/(t_2-t_1) \geq G'_+(t_1)\\
&\quad = \frac{1}{2}\left(f'_+\left((1-t_1)\frac{a+b}{2}+t_1 b\right)-f'_+\left(t_1 a+(1-t_1)\frac{a+b}{2}\right)\right)\left(\frac{b-a}{2}\right),
\end{aligned}$$

where $f'_+(x_0)$ denotes the right derivative of f in the point x_0.

By the convexity of f one gets:

$$f'_+\left((1-t_1)\frac{a+b}{2}+t_1 b\right) \geq f'_+\left(t_1 a+(1-t_1)\frac{a+b}{2}\right), \qquad t_1 \in (0,1),$$

which shows that G is monotonously increasing on $(0,1)$ and (see (ii)) also in $[0,1]$.

(ii) f being convex on $[a,b]$, we have:

$$G(t) \geq f\left(\frac{1}{2}\left(ta+(1-t)\frac{a+b}{2}+(1-t)\frac{a+b}{2}+tb\right)\right)=f\left(\frac{a+b}{2}\right),$$

which implies (4).

On the other hand, we also have:

$$\begin{aligned}
G(t) &\leq \frac{1}{2}\left(tf(a)+(1-t)f\left(\frac{a+b}{2}\right)+(1-t)f\left(\frac{a+b}{2}\right)+tf(b)\right)\\
&= t\cdot\frac{f(a)+f(b)}{2}+(1-t)f\left(\frac{a+b}{2}\right)
\end{aligned}$$

for all t in $[0,1]$, which implies that

$$G(t) \leq G(1) = \frac{f(a)+f(b)}{2}, \qquad t \in [0,1],$$

i.e., the statement (5).

(iii) Let us consider the mapping $g : [a,b] \to \mathbf{R}$, $g(x) = f\left(tx + (1-t)\frac{a+b}{2}\right)$. Clearly, g is convex on $[a,b]$ and by HADAMARD's inequality one has:

$$H(t) = \frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x \le \frac{g(a)+g(b)}{2} = G(t)$$

for all $t \in [0,1]$.

(iv) Since f is convex on $[(3a+b)/4,\ (a+3b)/4]$, HADAMARD's inequalities show the first part of (7). The same inequality applied for the convex mapping G yields the second part of (7) and we omit the details.

(v) f being differentiable convex on $[a,b]$, we have:

$$f\left(\frac{a+b}{2}\right) - f\left(tx + (1-t)\frac{a+b}{2}\right) \ge t\left(\frac{a+b}{2} - x\right) f'\left(tx + (1-t)\frac{a+b}{2}\right)$$

for al $t \in (0,1)$ and $x \in (a,b)$. Integrating this inequality over x on $[a,b]$ one gets:

$$f\left(\frac{a+b}{2}\right) - H(t) \ge \frac{t}{b-a}\int_a^b \left(\frac{a+b}{2} - x\right) f'\left(tx + (1-t)\frac{a+b}{2}\right)\mathrm{d}x.$$

Since

$$\frac{t}{b-a}\int \left(\frac{a+b}{2} - x\right) f'\left(tx + (1-t)\frac{a+b}{2}\right)\mathrm{d}x = H(t) - G(t), \quad t \in [0,1],$$

(8) is proven.

Now, we will consider another mapping associated to HADAMARD's inequality given by:

$$L : [0,1] \to \mathbf{R},\ L(t) := \frac{1}{2(b-a)}\int_a^b f\Big(ta + (1-t)x) + f((1-t)x + tb)\Big)\,\mathrm{d}x,$$

where $f : I \subseteq \mathbf{R} \to \mathbf{R}$ and $a, b \in I \quad (a < b)$.

The following theorem also holds

Theorem 3. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping on I and a, b are as above. Then*

(i) *L is convex on $[0,1]$;*

(ii) *We have the inequalities:*

$$G(t) \le L(t) \le \frac{1-t}{b-a}\int_a^b f(x)\,\mathrm{d}x + t \cdot \frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2} \tag{9}$$

for all $t \in [0,1]$ *and*

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}. \tag{10}$$

(iii) *One has the inequalities:*

$$H(1-t) \le L(t) \quad \textit{and} \quad \frac{H(t)+H(1-t)}{2} \le L(t) \tag{11}$$

for all $t \in [0,1]$.

Proof. (i) is obvious by the convexity of f (see e.g. [**4**]) and we shall omit the details.

(ii) By Jensen's integral inequality one has:

$$\begin{aligned} L(t) &\ge \frac{1}{2}\left(f\left(\frac{1}{b-a}\int_a^b ((1-t)x+ta)\,\mathrm{d}x\right) + f\left(\frac{1}{b-a}\int_a^b ((1-t)x+tb)\,\mathrm{d}x\right)\right) \\ &= \frac{1}{2}\left(f\left(ta+(1-t)\frac{a+b}{2}\right) + f\left(tb+(1-t)\frac{a+b}{2}\right)\right) = G(t). \end{aligned}$$

By the convexity of f one has:

$$\begin{aligned} L(t) &\le \frac{1}{2(b-a)}\int_a^b \Big((1-t)f(x)+tf(a)+(1-t)f(x)+tf(b)\Big)\mathrm{d}x \\ &= \frac{1-t}{b-a}\int_a^b f(x)\,\mathrm{d}x + t\cdot\frac{f(a)+f(b)}{2} \end{aligned}$$

for all $t \in [0,1]$.

The last part of (9) is obvious.

The bound (10) follows from (9).

By the convexity of f one has:

$$\begin{aligned} L(t) &\ge \frac{1}{b-a}\int_a^b f\left(\frac{ta+(1-t)x+(1-t)x+tb}{2}\right)\mathrm{d}x \\ &= \frac{1}{b-a}\int_a^b f\left((1-t)x+t\cdot\frac{a+b}{2}\right)\mathrm{d}x = H(1-t) \end{aligned}$$

for all $t \in [0,1]$ and the first part of (11) is proved. For the second part one has:

$$L(t) \ge H(1-t) \quad \text{and} \quad L(t) \ge G(t) \ge H(t)$$

for all $t \in [0,1]$.

Applications: 1. Let $p \geq 1$ and $0 \leq a < b$. Then one has the inequalities:

$$0 \leq \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} - \frac{1}{t(p+1)(b-a)}\left(\left(\frac{a+b}{2}+t\left(\frac{b-a}{2}\right)\right)^{p+1} - \left(\frac{a+b}{2}-t\left(\frac{b-a}{2}\right)\right)^{p+1}\right) \leq (1-t)\left(\frac{a^p+b^p}{2} - \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)$$

and

$$\frac{1}{t(p+1)(b-a)}\left(\left(\frac{a+b}{2}+t\left(\frac{b-a}{2}\right)\right)^{p+1} - \left(\frac{a+b}{2}-t\left(\frac{b-a}{2}\right)\right)^{p+1}\right) \leq \frac{1}{2}\left(\left(ta+(1-t)\frac{a+b}{2}\right)^p + \left((1-t)\frac{a+b}{2}+tb\right)^p\right)$$

for all $t \in (0,1]$.

The proof follows by the inequalities (2) and (6) applied to the convex mapping $f:[0,\infty)\to[0,\infty)$, $f(x)=x^p \quad (p \geq 1)$.

2. Let $0 < a < b$. Then one has:

$$0 \leq \frac{\ln b - \ln a}{b-a} - \frac{1}{t(b-a)} \ln\left(\frac{\frac{a+b}{2}+t\left(\frac{b-a}{2}\right)}{\frac{a+b}{2}-t\left(\frac{b-a}{2}\right)}\right) \leq (1-t)\left(\frac{a+b}{2ab} - \frac{\ln b - \ln a}{b-a}\right)$$

and

$$\frac{1}{t(b-a)} \ln\left(\frac{\frac{a+b}{2}+t\left(\frac{b-a}{2}\right)}{\frac{a+b}{2}-t\left(\frac{b-a}{2}\right)}\right) \leq \frac{1}{2}\,\frac{a+b}{\left(ta+(1-t)\frac{a+b}{2}\right)\left((1-t)\frac{a+b}{2}+tb\right)}$$

for all $t \in (0,1]$.

The proof is obvious from (2) and (6) for the mapping $f:(0,\infty)\to(0,\infty)$, $f(x) := 1/x$.

3. Let $p \geq 1$ and $0 \leq a < b$. Then one has the inequalities:

$$\frac{1}{2}\left(\left(ta+(1-t)\frac{a+b}{2}\right)^p + \left((1-t)\frac{a+b}{2}+tb\right)^p\right)$$
$$\leq \frac{1}{(1-t)(p+1)}\left(b^{p+1}+(ta+(1-t)b)^{p+1}-((1-t)a+tb)^{p+1}-a^{p+1}\right)$$
$$\leq \frac{1-t}{b-a}\cdot\frac{b^{p+1}-a^{p+1}}{p+1} + t\cdot\frac{a^p+b^p}{2} \leq \frac{a^p+b^p}{2}$$

for all $t \in [0,1)$.

The proof is obvious by the inequality (9) for $f(x)=x^p \quad (p \geq 1)$.

4. Let $0 < a < b$. Then one has:

$$\frac{\frac{a+b}{2}}{\left(ta+(1-t)\frac{a+b}{2}\right)\left((1-t)\frac{a+b}{2}+tb\right)} \le \frac{1}{2(b-a)(1-t)}\ln\left(\frac{b(ta+(1-t)b)}{a((1-t)a+tb)}\right)$$
$$\le \frac{1-t}{b-a}(\ln b - \ln a) + t\cdot\frac{a+b}{2ab} \le \frac{a+b}{2ab}$$

for all $t \in [0,1)$.

The proof is obvious by (9) for $f(x) = 1/x, \ \ x > 0$.

For other inequalities connected to HADAMARD's result we refer to [**1**–**6**] where further references are given.

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