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AN ASYMPTOTIC FORMULA INVOLVING THE ENUMERATING FUNCTION OF FINITE ABELIAN GROUPS

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Dedicated to Professor Paul Erdős on the occasion of his 80th birthday

Let $a(n)$ denote the number of non-isomorphic Abelian groups with n elements. It is shown that there is a constant $C > 0$ such that

$$\sum_{n \le x} a(n + a(n)) = Cx + O(x^{11/12+\varepsilon}).$$

Let $a(n)$, as usual, denote the number of non-isomorphic Abelian (commutative) groups with n elements. It is well-known that this is a multiplicative function ($a(mn) = a(m)a(n)$ if $(m, n) = 1$) such that $a(p^\alpha) = P(\alpha)$ for any prime p (henceforth p will always denote primes), where $P(\alpha)$ is the number of (unrestricted) partitions of α. Thus, for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\ldots ,$$

where $\zeta(s)$ is the RIEMANN zeta-function. The first paper in which the function $a(n)$ was studied was written by P. ERDŐS and G. SZEKERES [**5**]. Later the distribution of values of $a(n)$ was extensively studied by many authors (see, for example, the papers [**2**]–[**11**]). The function $a(n)$ has the property that $a(n) = a(s(n))$, where $s(n)$ is the squarefull part of n. Functions with this property were named s-functions in [**8**], where their local densities were discussed. Problems involving $a(n)$ at consecutive integers were investigated in [**3**], and those involving the iterates of $a(n)$ in [**4**]. The local densities of $a(n)$ were studied in [**6**], [**7**] and [**11**].

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The aim of this note is to furnish an asymptotic formula for the summatory function of $a(n+a(n))$. This is motivated by the work of C. Spiro **[12]**, who proved

$$\sum_{n\le x,\, d(n+d(n))=d(n)} 1 \gg \frac{x}{(\log x)^7},$$

where as usual $d(n)$ is the number of divisors of n. It seems reasonable to conjecture that, for some $D>0$,

$$\sum_{n\le x} d(n+d(n)) = Dx\log x + O(x), \tag{1}$$

although to the best of my knowledge no one has proved or disproved (1) so far. The corresponding problem when $d(n)$ is replaced by $a(n)$ (or a suitable prime--independent multiplicative function $f(n)$ such that $f(p)=1$) is much less difficult. This is roughly due to the fact that $d(p)=2$ and $a(p)=1$. We shall prove the following

Theorem. *There is an effectively computable constant $C>0$ such that for any given $\varepsilon>0$*

$$\sum_{n\le x} a(n+a(n)) = Cx + O(x^{11/12+\varepsilon}). \tag{2}$$

Before we give the proof of (2) it should be remarked that very likely the exponent 11/12 in the error term in (2) is far from the best possible one. In fact, I conjecture that the error term in question is $O(x^{1/2+\varepsilon})$ for any given $\varepsilon>0$ and $\Omega(x^{1/2-\delta})$ for any given $\delta>0$. It would be interesting to compare C in (2) with the constant

$$C_0 = \lim_{x\to\infty} x^{-1}\sum_{n\le x} a(n) = \zeta(2)\zeta(3)\zeta(4)\ldots = 2.29485\ldots\ ,$$

which represents the mean value of $a(n)$. Numerical computation of C is not easy, and I cannot rule out the possibility that $C=C_0$.

Henceforth let q denote squarefree numbers ($\mu^2(q)=1$) and s squarefull numbers ($p\mid s$ implies $p^2\mid s$), respectively. We start from

$$\sum_{n\le x} a(n+a(n)) = \sum_{k\le x^\varepsilon}\ \sum_{n\le x,\, a(n)=k} a(n+k) = \sum_{k\le x^\varepsilon} S(x,k),$$

where we used the bound (see **[9]**) $\log a(n) \ll \log n/\log\log n$, which implies $a(n)\le n^\varepsilon$ $(n\ge n_0)$, and where we set

$$\begin{aligned} S(x,k) := \sum_{n\le x,\, a(n)=k} a(n+k) &= \sum_{s\le x,\, a(s)=k}\ \sum_{q\le x/s,\, (q,s)=1} a(qs+k) \\ &= \sum_{s\le x^\alpha,\, a(s)=k}\ \sum_{q\le x/s,\, (q,s)=1} a(qs+k) + O(x^{1-\alpha/2+\varepsilon}) \end{aligned}$$

uniformly in k, where α is a constant such that $0 < \alpha < 1/3$ which will be determined later. Here we used the already mentioned property that $a(n) = a(s)$ if $n = qs$, $(q,s) = 1$, which follows from $a(p) = 1$ and multiplicativity. Now we shall use (see [**7**] and [**8**]) the uniform estimate

$$\sum_{q\le x,\,(q,r)=1} 1 = 6\pi^{-2}x\prod_{p|r}(1+p^{-1})^{-1} + O(x^{1/2}r^{\varepsilon})$$

to obtain, uniformly for $1 \le s \le x^{\alpha} < x^{1/3}$, $1 \le k \le x^{\varepsilon}$,

$$\sum_{q\le x/s,\,(q,s)=1} a(qs+k) = \sum_{d^2l\le x/s,\,(d,s)=1,\,(l,s)=1} \mu(d)a(d^2ls+k)$$

$$= \sum_{d\le x^{\alpha/2},\,(d,s)=1} \mu(d) \sum_{l\le x/(d^2s),\,(l,s)=1} a(d^2ls+k) + O\Bigl(\frac{x}{s}\sum_{d>x^{\alpha/2}} x^{\varepsilon}d^{-2}\Bigr)$$

$$= \sum_{d\le x^{\alpha/2},\,(d,s)=1} \mu(d)\sum_{\delta|s}\mu(\delta) \sum_{m\le x/(d^2\delta s)} a(d^2\delta sm+k) + O(x^{1+\varepsilon-\alpha/2}s^{-1})$$

$$= \sum_{d\le x^{\alpha/2},\,(d,s)=1} \mu(d)\sum_{\delta|s}\mu(\delta) \sum_{n\le x,\,n\equiv k \pmod r} a(n) + O(x^{\alpha/2+\varepsilon}) + O(x^{1+\varepsilon-\alpha/2}s^{-1}).$$

Here we set $r = d^2\delta s$ and we used the elementary relations

$$\mu^2(n) = \sum_{d^2|n}\mu(d), \sum_{d|n}\mu(d) = \begin{cases} 1 & n=1,\\ 0 & n>1.\end{cases}$$

Moreover the first O-term above may be absorbed by the second one if $0 < \alpha < 1/3$. Thus we are left with the evaluation of

$$T(x,k,r) := \sum_{n\le x,\,n\equiv k \pmod r} a(n) \qquad (1 \le k \le x^{\varepsilon},\ 1 \le r \le x),$$

and we seek an asymptotic formula for $T(x,k,r)$ with the error term uniform in k and r. There are results in the literature on $T(x,k,r)$ due to H.-E. Richert [**11**] and J. Duttlinger [**2**]. For $(k,r) = 1$ one can get by an elementary argument, uniformly for $1 \le r \le x$,

$$T(x,k,r) = B(r)x + O((rx)^{1/2+\varepsilon}), \qquad B(r) = O(1/r), \tag{3}$$

where $B(r)$ is given explicitly by (5) and (6). For $1 \le k \le x^{\varepsilon}$, $1 \le r \le x$, $(k,r) > 1$ formula (5.2) of Duttlinger's paper implies that uniformly

$$T(x,k,r) = B(k,r)x + O((rx)^{1/2+\varepsilon}), \qquad B(k,r) = O(r^{\varepsilon-1}), \tag{4}$$

since in that case $d = d_1d_2 \ll x^{\varepsilon}$ and (4) follows by the condition $\alpha(m) \mid d_1$, $\alpha(m) = \prod_{p|m} p$, so that one can majorize over squarefull numbers. Thus combining (3) and (4) we have a formula for $T(x,k,r)$ valid in all cases, where we set $B(r) = B(k,r)$ if $(k,r) = 1$ for notational convenience. If r is bounded by a (relatively) small power of x, then both Richert and Duttlinger obtain much sharper results for $T(x,k,r)$, which involve the existence of two more main terms besides $B(k,r)x$. An obvious way to improve on the exponent 11/12 in (2) is to sharpen (3) and (4). We postpone the proof of (3) and carry on with the evaluation of $S(x,k)$. We obtain

$$\begin{aligned} S(x,k) &= O(x^{1+\varepsilon-\alpha/2}) \\ &\quad + \sum_{s\le x^\alpha,\, a(s)=k} \quad \sum_{d\le x^{\alpha/2},\,(d,s)=1} \mu(d)\sum_{\delta|s}\mu(\delta)\bigl\{xB(k,d^2\delta s) + O(x^{1/2+\varepsilon}d(\delta s)^{1/2})\bigr\} \\ &= x\Bigl\{\sum_{s=1,\, a(s)=k} \quad \sum_{d=1,\,(d,s)=1} \mu(d)\sum_{\delta|s}\mu(\delta)B(k,d^2\delta s)\Bigr\} \\ &\quad + O(x^{1+\varepsilon-\alpha/2}) + O\Bigl(x^{1/2+\varepsilon}\sum_{s\le x^\alpha} s \sum_{d\le x^{\alpha/2}} d\Bigr) \\ &= xA(k) + O(x^{1+\varepsilon-\alpha/2}) + O(x^{(1+5\alpha)/2+\varepsilon}) = xA(k) + O(x^{11/12+\varepsilon}) \end{aligned}$$

for $\alpha = 1/6$ with

$$A(k) := \sum_{s=1,\, a(s)=k} \quad \sum_{d=1,\,(d,s)=1} \mu(d)\sum_{\delta|s}\mu(\delta)B(k,d^2\delta s).$$

By using $B(k,r) \ll r^{\varepsilon-1}$ in the relevant range it follows that

$$\begin{aligned} \sum_{n\le x} a(n+a(n)) &= \sum_{k\le x^\varepsilon} S(x,k) = x\sum_{k\le x^\varepsilon} A(k) + O(x^{11/12+\varepsilon}) \\ &= x\sum_{k=1}^\infty A(k) + O\Bigl(x\sum_{k>x^\varepsilon} \quad \sum_{s=1,\, a(s)=k} s^{\varepsilon-1}\Bigr) + O(x^{11/12+\varepsilon}) \\ &= x\sum_{k=1}^\infty A(k) + O(x^{11/12+\varepsilon}), \end{aligned}$$

since $a(s) = k$ implies $s \gg \exp(C_1 \log k/\log\log k)$ for some $C_1 > 0$, and $\sum_{s\le x} 1 \ll x^{1/2}$. This proves (2) with

$$C = \sum_{k=1}^\infty A(k) > 0.$$

Finally it remains to sketch the proof of (3) for $(k,r) = 1$. Let $\chi(n)$ be a Dirichlet character modulo r. If χ_0 is the principal character mod r, then

$$\begin{aligned} \sum_{n\le x} \chi_0(n)a(n) &= \sum_{n\le x,\,(n,r)=1} a(n) = \sum_{s\le x,\,(s,r)=1} a(s) \sum_{q\le x/s,\,(q,s)=1,\,(q,r)=1} 1 \\ &= \sum_{s\le x,\,(s,r)=1} a(s)\Bigl\{\frac{6x}{\pi^2 s}\prod_{p|sr}\frac{1}{1+p^{-1}} + O\Bigl(\Bigl(\frac{x}{s}\Bigr)^{1/2} x^\varepsilon\Bigr)\Bigr\} \\ &= \frac{6x}{\pi^2}\prod_{p|r}\frac{1}{1+p^{-1}}\prod_{p\nmid r}\Bigl(1 + \frac{a(p^2)p^{-2} + a(p^3)p^{-3} + \cdots}{1+p^{-1}}\Bigr) + O(x^{1/2+\varepsilon}) \end{aligned}$$

$$= 6\pi^{-2}C(r)x + O(x^{1/2+\varepsilon})$$

uniformly for $1 \le r \le x$, where we have set

$$C(r) := \prod_{p|r} \frac{1}{1+p^{-1}} \prod_{p\nmid r}\left(1 + \frac{P(2)p^{-2} + P(3)p^{-3} + \cdots}{1+p^{-1}}\right) = O(1). \tag{5}$$

If $\chi(n)$ is a non-principal character mod r, then

$$\sum_{n\le x} a(n)\chi(n) = \sum_{s\le x}\chi(s)a(s) \sum_{q\le x/s,\,(q,s)=1} \chi(q)$$
$$= \sum_{s\le x}\chi(s)a(s) \sum_{d^2l\le x/s,\,(d,s)=1,\,(l,s)=1} \mu(d)\chi(d^2l)$$
$$= \sum_{s\le x}\chi(s)a(s) \sum_{d\le (x/s)^{1/2},\,(d,s)=1} \mu(d)\chi^2(d) \sum_{l\le x/(d^2s),\,(l,s)=1} \chi(l)$$
$$= \sum_{s\le x}\chi(s)a(s) \sum_{d\le (x/s)^{1/2},\,(d,s)=1} \mu(d)\chi^2(d) \sum_{\delta|s}\mu(\delta)\chi(\delta) \sum_{m\le x/(d^2\delta s)} \chi(m).$$

To estimate the innermost sum above we use the classical Pólya-Vinogradov inequality (see H. Davenport [**1**, Ch. 23]):

$$\sum_{M<n\le M+N} \chi(n) = O(r^{1/2}\log r) \qquad (\chi \ne \chi_0;\ M, N \ge 1).$$

This gives

$$\sum_{n\le x} a(n)\chi(n) \ll x^{1/2}\sum_{s\le x} a(s)d(s)s^{-1/2}r^{1/2}\log r \ll (xr)^{1/2+\varepsilon}.$$

Using then the orthogonality relations for characters (φ is Euler's function), namely

$$\frac{1}{\varphi(r)} \sum_{\chi \pmod r} \chi(n)\bar\chi(k) = \begin{cases} 1 & n \equiv k \pmod r \\ 0 & n \not\equiv k \pmod r \end{cases} \qquad ((k,r)=1)$$

we obtain

$$T(x,k,r) = \frac{1}{\varphi(r)} \sum_{\chi \pmod r} \bar\chi(k)\Big(\sum_{n\le x}\chi(n)a(n)\Big)$$
$$= \frac{6C(r)x}{\pi^2\varphi(r)} + O(x^{1/2+\varepsilon}) + \frac{1}{\varphi(r)} \sum_{\chi \pmod r,\,\chi\ne\chi_0} \bar\chi(k)\Big(\sum_{n\le x}\chi(n)a(n)\Big)$$
$$= B(r)x + O\big((xr)^{1/2+\varepsilon}\big)$$

uniformly in r, where

$$(6)\qquad \begin{aligned} B(r) &:= \frac{6C(r)}{\pi^2\varphi(r)} = \\ &= \frac{6}{\pi^2 r}\prod_{p|r}\Big(1-\frac{1}{p}\Big)^{-1}\Big(1+\frac{1}{p}\Big)^{-1}\prod_{p\nmid r}\Big(1+\frac{P(2)p^{-2}+P(3)p^{-3}+\cdots}{1+p^{-1}}\Big) \\ &= O\Big(\frac{1}{r}\prod_{p|r}\frac{1}{1-p^{-2}}\Big) = O\Big(\frac{1}{r}\Big). \end{aligned}$$

This proves (3), and completes the proof of the Theorem.

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