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A CONTRIBUTION TO THE STUDY OF REAL GRAPH POLYNOMIALS

Ivan Gutman

Let G be a graph and C its circuit. Let α stand for the matching polynomial. Let t be a real number, such that $|t| \leq 1$. We demonstrate that for certain classes of compound graphs the polynomial $\beta_t(G, C, x) = \alpha(G, x) - 2t\,\alpha(G - C, x)$ is real, i.e. that all its zeros are real.

1. INTRODUCTION

Let G be a graph containing n vertices and m edges. For $k > 1$ let $m(G, k)$ denote the number of ways in which k independent edges can be selected in G. In addition to this, let $m(G, 0) = 1$ and $m(G, 1) = m$. Then the matching polynomial of G is defined as

$$\alpha(G) = \alpha(G, x) = \sum_{k \geq 0} (-1)^k m(G, k) x^{n-2k}.$$

The theory of the matching polynomials is nowadays well elaborated [**2**]. In particular, it is known that $\alpha(G)$ is a real graph polynomial [**4**], i.e. that all its zeros are real-valued numbers.

Consider now another polynomial defined as

$$\beta(G, C) = \beta(G, C, x) = \alpha(G, x) - 2\alpha(G - C, x)$$

where C is a circuit of the graph G and $G - C$ is the subgraph obtained by deleting all the vertices of C from G; if C is a Hamiltonian circuit of G, then $\alpha(G - C) \equiv 1$.

The zeros of the polynomial β play a distinguished role in AIHARA's theory of cyclic conjugation [**1**], [**12**]. It has been recently conjectured [**8**] that the polynomial $\beta(G, C)$ is real for all cyclic graphs G and all circuits C contained in them. The conjecture was supported by proving its validity for various classes of graphs [**7**], [**8**], [**11**], [**12**].

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In the case of unicyclic graph $\beta(G, C)$ coincides with the characteristic polynomial. Hence we have the following elementary result.

Observation 1. *If G is a unicyclic graph, then $\beta(G, C)$ is a real polynomial.*

The above observation may serve as a motivation for a generalization of the β-polynomial concept. Define

$$\beta_t(G, C, x) = \beta_t(G, C) = \alpha(G, x) - 2t\alpha(G - C, x).$$

Observation 2. *If G is a unicyclic graph and $-1 \leq t \leq 1$, then $\beta_t(G, C)$ is a real polynomial.*

This result follows from the fact that it is possible to construct a weighted digraph G^t whose characteristic polynomial coincides with $\beta_t(G, C)$. In order to do this, an edge of G belonging to the circuit C is to be exchanged by a pair of oppositely directed arcs, having weights $e^{i\theta}$ and $e^{-i\theta}$. The characteristic polynomial of G^t obeys the relation [**6**], [**9**], [**12**]

$$\phi(G^t) = \alpha(G) - [e^{i\theta} + e^{-i\theta}]\alpha(G - C) = \alpha(G) - 2t\alpha(G - C), \tag{1}$$

where $t = \cos\theta$. For θ being a real number the adjacency matrix of G^t is Hermitean and therefore all the zeros of $\phi(G^t)$ are real-valued.

Examples show that when the parameter t is less than -1 or greater than $+1$, then the β_t-polynomial of a unicyclic graph may have complex-valued zeros. The simplest such example is provided by the triangle, whose β_t-polynomial is $x^3 - 3x - 2t$. It is easy to verify that all the three zeros of this polynomial are real if and only if $|t| \leq 1$.

The result formulated here as Observation 2 can easily be extended to graphs possessing several circuits, such that no two circuits share a common edge [**6**], [**9**]:

Observation 3. *If G is a graph whose no edge belongs to more than one circuit and $-1 \leq t \leq 1$, then for all circuits C of G, $\beta_t(G, C)$ is a real polynomial.*

In what follows we show that the β-polynomials of some other polycyclic graphs are also real for all values of t, $|t| \leq 1$.

2. THE COMPOUND GRAPHS G_n AND $G_n[]$

The one-vertex graph will be denoted by K_1 and its vertex labeled by u_0.

Let H be an n-vertex graph and let S be a subset of its vertex set. Then by $H[S]$ we denote the $(n + 1)$-vertex graph obtained by connecting all elements of S with the vertex u_0 of K_1.

Let G be a graph and v and w its two (not necessarily distinct) vertices. Construct the graph G_n by taking n copies $(n > 1)$ of G and joining the vertex v

Fig. 1

of the $(j+1)$-th copy to the vertex w of the j-th copy, $j = 1, 2, \ldots, n-1$ and, in addition, the vertex v of the first copy to the vertex w of the n-th copy (see Fig. 1).

The compound graph G_n has the following obvious property: The deletion from G_n of any vertex labeled by v results in the same subgraph; this subgraph will be denoted by $G_n - v$.

If G is a tree then G_n is unicyclic and its unique circuit will be denoted by C^*. Since all the vertices of G_n, labeled by v, necessarily belong to the circuit C^*, the subgraph $G_n - v$ is acyclic (but needs not be connected).

Denote by $_n$ and $_n$ the sets of vertices of G_n labeled by v and w, respectively (see Fig. 1). The cardinalities of these sets are, of course, equal to n.

Lemma 1. *For any* $\subseteq _n$,

$$\alpha(G_n[]) = x\alpha(G_n) - ||\alpha(G_n - v) \tag{2}$$

where $||$ *stands for the cardinality of the set .*

If $= \varnothing$ then $G_n[]$ is isomorphic to the disconnected graph $G_n \cup K_1$. Because of the identity [**2**], [**5**]

$$\alpha(H_a \cup H_b) = \alpha(H_a)\alpha(H_b) \tag{3}$$

and the fact that $\alpha(K_1) = x$, in the case when is the empty set, eq. (1) is satisfied in a trivial manner.

It remains, therefore, to examine only the case when the set is non-empty.

If e is an edge of the graph H, connecting the vertices p and q, then [**2**], [**5**],

$$\alpha(H) = \alpha(H - e) - \alpha(H - p - q). \tag{4}$$

Applying the recurrence relation (4) to an edge of $G_n[]$, connecting a vertex from with u_0, we obtain

$$\alpha(G_n[]) = \alpha(G_n[]) - \alpha(G_n - v)$$

where $' = \setminus \{v\}$. If $'$ is non-empty, then the application of (4) has to be repeated to the edge of the graph $G_n[']$, connecting u_0 and v', $v' \in '$. This yields

$$\alpha(G_n[]) = \alpha(G_n['']) - 2\alpha(G_n - v)$$

where $'' = \setminus \{v, v'\}$. If $''$ is non-empty then the procedure has to be repeated again, etc. Ultimately we arrive at

$$\alpha(G_n[]) = \alpha(G_n \cup K_1) - ||\alpha(G_n - v).$$

Lemma 1 follows now from eq. (3). □

In a fully analogous manner we may deduce

Lemma 2. *Let $G - v$ and $G - w$ be isomorphic graphs and $v \neq w$. Then for any* $\subseteq_n$ *and* $\subseteq_n$

$$\alpha(G_n[\cup]) = x\alpha(G_n) - (|| + ||)\alpha(G_n - v).$$

According to Lemmas 1 and 2 the matching polynomials of $G_n[]$ and $G_n[\cup]$ are independent of the actual choice of the vertices to which the vertex u_0 is connected and depend only on their number, i.e. on the cardinalities of the sets and .

3. AN AUXILIARY WEIGHTED DIGRAPH

Denote by $\phi(H) = \phi(H, x)$ the characteristic polynomial of the graph H. Suppose that the vertices p and q of H are connected by an edge e and that the weight of this edge is k. A well known result of graph spectral theory [**3**] is the relation

$$\phi(H) = \phi(H - e) - k^2\phi(H - p - q), \tag{5}$$

which holds provided e is a bridge. (Recall that an edge e is said to be a bridge of the graph H if $H - e$ has more components than H.) Another well known identity for the characteristic polynomial is [**3**]

$$\phi(H_a \cup H_b) = \phi(H_a)\phi(H_b). \tag{6}$$

Let ^{1} be a one-element subset of $_n$. Then $G_n[^1]^k$ denotes the graph obtained from $G_n[^1]$ by associating the weight k to the edge e_0 which connects the vertex u_0 with the vertex $v \in {}^1$. Observe that e_0 is a bridge.

As already pointed out, if G is a tree then G_n is unicyclic and its unique circuit is denoted by C^*. If G is a tree then $G_n^t[^1]^k$ denotes the digraph obtained from $G_n[^1]^k$ by exchanging an edge (any edge) belonging to C^* by a pair of oppositely directed arcs, having weights $e^{i\theta}$ and $e^{-i\theta}$, $t = \cos\theta$.

Applying eq. (5) to the edge e_0 of $G_n^t[^1]^k$ and bearing in mind eq. (6) as well as $\phi(K_1) = x$, we immediately arrive at

Lemma 3. *If G is a tree then*

$$\phi(G_n^t[^1]^k) = x\phi(G_n^t) - k^2\phi(G_n - v).$$

Further, as a special case of eq. (1) we have

Lemma 4. *If G is a tree then*

$$\phi(G_n^t) = \alpha(G_n) - 2t\alpha(G_n - C^*).$$

4. THE MAIN RESULTS

Theorem 1. *If G is a tree, then for all t, $-1 \le t \le 1$, all $n > 1$ and all $\subseteq {}_n$,*

$$\beta_t(G_n[], C^*) = \phi(G_n^t[^1]^k)$$

with $k = \sqrt{||}$.

Theorem 2. *Let $G - v$ and $G - w$ be isomorphic graphs and $v \neq w$. If G is a tree, then for all t, $-1 \le t \le 1$, all $n > 1$, all $\subseteq {}_n$ and all $\subseteq {}_n$,*

$$\beta_t(G_n[\cup], C^*) = \phi(G_n^t[^1]^k)$$

with $k = \sqrt{|| + ||}$.

From the definition of the β_t-polynomial,

$$\beta_t(G_n[], C^*) = \alpha(G_n[]) - 2t\alpha(G_n - C^* \cup K_1).$$

Using eq. (3) and the fact that $\alpha(K_1) = x$ we get

$$\beta_t(G_n[], C^*) = \alpha(G_n[]) - 2xt\alpha(G_n - C^*)$$

which combined with Lemma 1 yields

$$\beta_t(G_n[], C^*) = x[\alpha(G_n) - 2t\alpha(G_n - C^*)] - ||\alpha(G_n - v). \tag{7}$$

If a graph H is acyclic, then [**2**], [**5**], $\phi(H) \equiv \alpha(H)$. Consequently, $\alpha(G_n - v) \equiv \phi(G_n - v)$. Bearing this fact in mind and using Lemma 4, the right-hand side of (7) is readily transformed into

$$\beta_t(G_n[], C^*) = x\phi(G_n^t) - ||\alpha(G_n - v).$$

Theorem 1 follows now from Lemma 3. □

Proof of Theorem 2 is analogous, except that instead of Lemma 1 we now have to employ Lemma 2. □

All the zeros of the characteristic polynomial of the auxiliary weighted digraph $G_n^t[^1]^k$ are real-valued [**3**]. Therefore we have

Corollary 1.1. *Under the conditions specified in Theorem 1, $\beta_t(G_n[])$ is a real polynomial.*

Corollary 2.1. *Under the conditions specified in Theorem 2, $\beta_t(G_n[\cup])$ is a real polynomial.*

Theorems 1 and 2 can be further extended. Let the graphs $G_n[]T$ and $G_n^t[^1]^kT$ be obtained by identifying the vertex u_0 of $G_n[]$ and $G_n^t[^1]^k$, respectively, with the root of a rooted tree T. Then we can prove the following results.

Theorem 3. *Let T be an arbitrary rooted tree. Theorem 1 remans valid if $G_n[]$ and $G_n^t[^1]^k$ are exchanged by $G_n[]T$ and $G_n^t[^1]^kT$, respectively.*

Theorem 4. *Let T be an arbitrary rooted tree. Theorem 2 remains valid if $G_n[\cup]$ and $G_n^t[^1]^k$ are exchanged by $G_n[\cup]$ and $G_n^t[^1]^k T$, respectively.*

Theorems 3 and 4 imply that under conditions specified in Theorems 1 and 2, $\beta_t(G_n[]T, C^*)$ and $\beta_t(G_n[\cup]T, C^*)$ are real polynomials for all rooted trees T and for all values of the parameter t, $-1 \leq t \leq 1$.

Remark. The special case of Theorems 1 and 3, when the vertices v and w coincide and when $t = 1$ was previously reported by the author in [**7**].

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Faculty of Science,
University of Kragujevac,
P.O. Box 60, 34000 Kragujevac,
Yugoslavia

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