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ON A MODIFIED BIRKHOFF-YOUNG QUADRATURE FORMULA FOR ANALYTIC FUNCTIONS

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Earlier D. Đ. Tošić derived a modification of the Birkhoff-Young quadrature formula for analytic functions, where the error term R_{MF} is given as an infinite series. In this paper a direct proof of the modified formula is given where the error term appears in integral form.

1. INTRODUCTION

In [**1**] Birkhoff and Young derived the five-point interpolation formula

$$(1)\qquad \int\limits_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z = \frac{8}{5}hf(z_0) + \frac{4h}{15}\big(f(z_0+h)+f(z_0-h)\big) - \frac{h}{15}\big(f(z_0+ih)+f(z_0-ih)\big) + R_{BY},$$

where f is analytic in a region D which contains the line-segment of integration, and the error term R_{BY} vanishes on polynomials of degree 5 or less. In [**2**] D. Đ. Tošić obtained a modified version of (1), namely

$$(2)\qquad \int\limits_{z_0-h}^{z_0+h} f(z)\,\mathrm{d}z = \frac{16}{15}hf(z_0) + \frac{h}{6}\Big[7/5+\sqrt{7/3}\Big]\ \Big[f\big(z_0+h\sqrt[4]{3/7}\big)+f\big(z_0-h\sqrt[4]{3/7}\big)\Big] + \frac{h}{6}\Big[7/5-\sqrt{7/3}\Big]\ \Big[f\big(z_0+ih\sqrt[4]{3/7}\big)+f\big(z_0-ih\sqrt[4]{3/7}\big)\Big] + R_{MF},$$

1991 AMS Subject Classification: 65 E 05 where the error term R_{MF} is given as an infinite series

$$R_{MF} = \frac{h^9}{793\,800} f^{(8)}(z_0) + \frac{h^{11}}{61\,122\,600} f^{(10)}(z_0) + \cdots,$$

and integration is taken along the line-segment with end-points $z_0 - h$ and $z_0 + h$. Using $\int_{-1}^{1} e^x \,\mathrm{d}x$, D. Đ. Tošić in [**2**] compares the Birkhoff-Young five-point formula, the three-point Gauss-Legendre formula, and the five-point modified Birkhoff-Young formula. It would appear that the modified Birkhoff-Young formula gives the greatest accuracy in this case.

In this note we give an elementary derivation of the modified Birkhoff-Young formula (2), and the error term R_{MF} now appears in integral form. As in [**2**] we begin by showing that

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = 2\left(1 - \frac{1}{5k^4}\right) f(0) + \left(\frac{1}{6k^2} + \frac{1}{10k^4}\right)\big(f(k) + f(-k)\big) + \left(-\frac{1}{6k^2} + \frac{1}{10k^4}\right)\big(f(ki) + f(-ki)\big) + R, \tag{3}$$

but now the error term R appears in integral form.

2. DERIVATION OF THE MODIFIED BIRKHOFF-YOUNG FORMULA

Using Cauchy's integral formula we have immediately

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = \int_{-1}^{1} \frac{1}{2\pi i} \oint_C \frac{f(t)}{t - z}\,\mathrm{d}t\,\mathrm{d}z,$$

where C is a positively-oriented simple contour with the line-segment of integration lying inside C. Using the algebraic identity

$$\frac{1}{t - z} = \frac{1}{t} + \frac{z}{t^2} + \frac{z^2}{t^3} + \frac{z^3}{t^4} + \frac{z^4}{t^5} + \frac{z^5}{t^5(t - z)},$$

and interchanging the order of integration, we have

$$\begin{aligned}\int_{-1}^{1} f(z)\,\mathrm{d}z &= \frac{1}{2\pi i} \oint_C f(t) \int_{-1}^{1} \left(\frac{1}{t} + \frac{z}{t^2} + \cdots + \frac{z^4}{t^5}\right) \mathrm{d}z\,\mathrm{d}t + \frac{1}{2\pi i} \int_{-1}^{1} \oint_C \frac{z^5 f(t)}{t^5(t - z)}\,\mathrm{d}t\,\mathrm{d}z \\ &= \frac{1}{\pi i} \oint_C f(t) \left(\frac{1}{t} + \frac{1}{3t^3} + \frac{1}{5t^5}\right) \mathrm{d}t + R_1,\end{aligned}$$

giving

$$\int_{-1}^{1} f(z)\,\mathrm{d}z = 2f(0) + \frac{1}{\pi i}\oint_C \left(\frac{1}{3t^3} + \frac{1}{5t^5}\right) f(t)\,\mathrm{d}t + R_1, \tag{4}$$

where

$$R_1 = \frac{1}{2\pi i}\int_{-1}^{1}\oint_C \frac{z^5 f(t)}{t^5(t-z)}\,\mathrm{d}t\,\mathrm{d}z.$$

With $\alpha = k$ and $\alpha = ik$ in the algebraic identity

$$\frac{1}{t-\alpha} + \frac{1}{t+\alpha} = \frac{2}{t} + \frac{2\alpha^2}{t^3} + \frac{2\alpha^4}{t^3(t^2-\alpha^2)}$$

we deduce that

$$\begin{aligned} f(k) + f(-k) - f(ik) - f(-ik) &= \frac{1}{2\pi i}\oint_C \left(\frac{1}{t-k} + \frac{1}{t+k} - \frac{1}{t-ik} - \frac{1}{t+ik}\right) f(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi i}\oint_C \frac{4k^2}{t^3} f(t)\,\mathrm{d}t + \frac{1}{2\pi i}\oint_C \frac{4k^6}{t^3(t^4-k^4)} f(t)\,\mathrm{d}t. \end{aligned}$$

We assume, of course, that the point $\pm k$, $\pm ik$ lie inside C. Rearranging terms we have

$$\frac{1}{\pi i}\oint_C \frac{f(t)}{3t^3}\,\mathrm{d}t = \frac{1}{6k^2}\big(f(k) + f(-k) - f(ik) - f(-ik)\big) + R_2, \tag{5}$$

where

$$R_2 = -\frac{k^4}{3\pi i}\oint_C \frac{f(t)}{t^3(t^4-k^4)}\,\mathrm{d}t.$$

Similarly, with $\alpha = k$ and $\alpha = ik$ in the algebraic identity

$$\frac{1}{t-\alpha} + \frac{1}{t+\alpha} = \frac{2}{t} + \frac{2\alpha^2}{t^3} + \frac{2\alpha^4}{t^5} + \frac{2\alpha^6}{t^5(t^2-\alpha^2)}$$

we deduce that

$$\begin{aligned} f(k) + f(-k) + f(ik) + f(-ik) &= \frac{1}{2\pi i}\oint_C \left(\frac{1}{t-k} + \frac{1}{t+k} + \frac{1}{t-ik} + \frac{1}{t+ik}\right) f(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi i}\oint_C \left(\frac{4}{t} + \frac{4k^4}{t^5} + \frac{4k^8}{t^5(t^4-k^4)}\right) f(t)\,\mathrm{d}t \end{aligned}$$

and hence, on rearranging terms, we get

$$\frac{1}{\pi i}\oint_C \frac{f(t)}{5t^5}\,\mathrm{d}t = \frac{1}{10k^4}\big(f(k) + f(-k) + f(ik) + f(-ik) - 4f(0)\big) + R_3, \tag{6}$$

where

$$R_3 = -\frac{k^4}{5\pi i} \oint_C \frac{f(t)}{t^5(t^4 - k^4)}\,\mathrm{d}t.$$

Using (4), together with (5) and (6), we get (3) with error term $R = R_1 + R_2 + R_3$, that is

$$R = \frac{1}{2\pi i} \int_{-1}^{1} \oint_C \frac{z^5\, f(t)}{t^5(t-z)}\,\mathrm{d}t\,\mathrm{d}z - \frac{k^4}{15\pi i} \oint_C \frac{(5t^2+3)f(t)}{t^5(t^4-k^4)}\,\mathrm{d}t.$$

Setting $k = \sqrt[4]{3/7}$ produces the modified Birkhoff-Young formula (2) in the case $z_0 = 0$ and $h = 1$, with the remainder

$$R_{MF} = \frac{1}{2\pi i} \int_{-1}^{1} \oint_C \frac{z^5 f(t)}{t^5(t-z)}\,\mathrm{d}t\,\mathrm{d}z - \frac{1}{35\pi i} \oint_C \frac{(5t^2+3)f(t)}{t^5\bigl(t^4 - \frac{3}{7}\bigr)}\,\mathrm{d}t.$$

By replacing $f(t)$ by $hf(z_0 + ht)$, it is a simple task to obtain formula (2) with the remainder R_{MF} in integral form.

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