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CONTINUOUS LINEAR FUNCTIONALS AND NORM DERIVATIVES IN REAL NORMED SPACES

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Some approximation theorems for the continuous linear functionals on real normed linear spaces in terms of norm derivatives are given.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives:

$$(x, y)_{i(s)} := \lim_{t \to 0-(+)} \frac{\|y + tx\|^2 - \|y\|^2}{2t}, \qquad \text{for all } x, y \in X.$$

For the sake of completeness we list some usual properties of these semi-inner products that will be used in the sequel:

— $(x, x)_p = \|x\|^2$ for all x in X;

— $(-x, y)_s = (x, -y)_s = -(x, y)_i$, if x, y are in X;

— $(\alpha x, \beta y)_p = \alpha\beta(x, y)_p$ for all x, y in X and $\alpha\beta \geq 0$;

— $(\alpha x + y, x)_p = \alpha(x, x)_p + (y, x)_p$ if x, y belong to X and α is in $\mathbb{R}$;

— $(x + y, z)_p \leq \|x\|\,\|z\| + (y, z)_p$ for all x, y, z in X;

— the element x in X is BIRKHOFF orthogonal over y in X, i.e., $\|x + ty\| \geq \|x\|$ for all t in $\mathbb{R}$ iff $(y, x)_i \leq 0 \leq (y, x)_s$, we denote $x \perp y\,(B)$;

— the space $(X, \|\cdot\|)$ is smooth iff $(y, x)_i = (y, x)_s$, for all x, y in X or iff $(\ ,\ )_p$ is linear in the first variable;

where $p = s$ or $p = i$.

For other properties of $(\ ,\ )_p$ in connection to the best approximation element or continuous linear functionals see [**2**] where further references are given.

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2. A CHARACTERIZATION OF REFLEXIVITY

The following theorem of R. C. James [**3**] is well-known:

Theorem 1. *Let X be a Banach space. X is reflexive if and only if for every closed and homogeneous hyperplane H there exists a point x in X, $x \neq 0$, such that x is Birkhoff orthogonal over H (we denote $x \perp H\,(B)$).*

The next theorem improves this result for real spaces.

Theorem 2. *Let X be a Banach space. X is reflexive if and only if for every continuous linear functional f on X there exists an element u in X such that the following estimation holds*:

$$(x,u)_i \le f(x) \le (x,u)_s \qquad \text{for all } x \in X. \tag{1}$$

Let H be a closed and homogeneous hyperplane in X and $f : X \to \mathbb{R}$ be a continuous linear functional on X such that $H = \mathrm{Ker}(f)$. Then from (1) it follows that $u \perp H\,(B)$ and by James's theorem, we conclude that X is reflexive.

Now, assume that X is reflexive and let f be a nonzero continuous linear functional on it. Since $\mathrm{Ker}(f)$ is a closed and homogeneous hyperplane in X there exists, by James's theorem, a nonzero element w_0 in X so that:

$$(x,w_0)_i \le 0 \le (x,w_0)_s \qquad \text{for all } x \in \mathrm{Ker}(f). \tag{2}$$

Because $f(x)w_0 - f(w_0)x \in \mathrm{Ker}(f)$ for all x in X, from (2) we derive that:

$$\big(f(x)w_0 - f(w_0)x, w_0\big)_i \le 0 \le \big(f(x)w_0 - f(w_0)x, w_0\big)_s \tag{3}$$

for all x in X and since

$$\big(f(x)w_0 - f(w_0)x, w_0\big)_p = f(x)\|w_0\|^2 - \big(x, f(w_0)w_0\big)_q, \qquad x \in X$$

where $p \neq q$, $p,q \in \{i,s\}$, we conclude, by (3), that

$$\left(\frac{x, f(w_0)w_0}{\|w_0\|^2}\right)_i \le f(x) \le \left(\frac{x, f(w_0)w_0}{\|w_0\|^2}\right)_s, \qquad x \in X$$

from where results (1) with $u := f(w_0)w_0/\|w_0\|^2$. This completes the proof.

The following corollary holds (see also [**4**]):

Corollary. *Let X be a Banach space. Then the following statements are equivalent*:

i) *X is reflexive and smooth*;

ii) *for every continuous linear functional $f : X \to \mathbb{R}$ there exists an element u in X such that*:

$$f(x) = (x,u)_s \qquad \text{for all } x \in X. \tag{4}$$

Remark 1. If f satisfies (1) or (4) then $\|f\| = \|u\|$ and $f(u) = \|u\|^2$. The proof of this fact is obvious and we shall omit the details.

3. APPROXIMATION OF CONTINUOUS LINEAR FUNCTIONALS

The following approximation of continuous linear functionals in terms of norm derivatives is valid:

Theorem 3. *Let X be a real normed linear space and f be a nonzero continuous linear functional on it. Then for every $\varepsilon > 0$ there exists a nonzero element x_ε in X and a positive number r_ε so that:*

(5) $$|f(x) - (x, x_\varepsilon)_p| \leq \varepsilon \qquad \textit{for all } x \in \bar{B}(0, r_\varepsilon) \textit{ and } p \in \{i, s\},$$

where $\bar{B}(0, r_\varepsilon)$ is the closed ball $\{x \in X \mid \|x\| \leq r_\varepsilon\}$.

Let $\varepsilon > 0$. Then there exists a nonzero element y_ε in X such that $\|y_\varepsilon\| = \varepsilon$ and $y_\varepsilon \notin \mathrm{Ker}(f)$.

On the other hand, we have:

$$|(y, y_\varepsilon)_s| \leq \|y\|\,\|y_\varepsilon\| = \varepsilon\|y\| \qquad \text{for all } \; y \in \mathrm{Ker}(f).$$

Now, for all x in X we have $y := f(x)y_\varepsilon - f(y_\varepsilon)x \in \mathrm{Ker}(f)$, and by the above inequality, we deduce that:

$$\left|\big(f(x)y_\varepsilon - f(y_\varepsilon)x, y_\varepsilon\big)_s\right| \leq 2\varepsilon^2\|f\|\,\|x\| \qquad \text{for all } \; x \in X.$$

On the other hand, a simple calculation shows that

$$\big(f(x)y_\varepsilon - f(y_\varepsilon)x, y_\varepsilon\big)_s = f(x)\|y_\varepsilon\|^2 - \big(x, f(y_\varepsilon)y_\varepsilon\big)_i,$$

for all x in X and then the above inequality becomes:

$$\left|f(x) - \left(\frac{x, f(y_\varepsilon)y_\varepsilon}{\|y_\varepsilon\|^2}\right)_i\right| \leq 2\|f\|\,\|x\| \qquad \text{for all } \; x \in X.$$

Putting $x_\varepsilon := f(y_\varepsilon)y_\varepsilon/\|y_\varepsilon\|^2 \neq 0$ and $r_\varepsilon := \varepsilon/2\|f\| > 0$ we obtain the estimation (5) for $p = i$. Now, it is obvious that if we replace x by $-x$, then (5) holds for $p = s$ too. This completes the proof.

Now, we shall introduce a definition.

Definition 1. *A nonzero continuous linear functional f defined on real normed space X is said to be of (APP)-type if for any $\varepsilon \in (0, 1)$ there exists a nonzero element y_ε in X such that:*

(6) $$|(y, y_\varepsilon)_p| \leq \varepsilon\|y\|\,\|y_\varepsilon\| \qquad \textit{for all } \; y \in \mathrm{Ker}(f),$$

where $p = s$ or $p = i$.

Remark 2. Clearly, y is not in $\mathrm{Ker}(f)$ and

$$|(y, y_\varepsilon)_i| \leq \varepsilon\|y\|\,\|y_\varepsilon\| \qquad \text{for all } \; y \in \mathrm{Ker}(f)$$

if and only if:

$$|(y, y_\varepsilon)_s| \leq \varepsilon\|y\|\,\|y_\varepsilon\| \qquad \text{for all } \; y \in \mathrm{Ker}(f)$$

where $\varepsilon \in (0, 1)$.

The following result improves Theorem 3.

Theorem 4. *Let f be a nonzero continuous linear functional of (APP)-type. Then for any $\varepsilon > 0$ there exists a nonzero element x_ε in X such that:*

$$|f(x) - (x, x_\varepsilon)_p| \le \varepsilon \|x\| \quad \text{for all } x \in X \tag{7}$$

and for any $p \in \{s, i\}$.

Since f is nonzero it follows that $\mathrm{Ker}(f)$ is closed in X and $\mathrm{Ker}(f) \neq X$. Let $\varepsilon > 0$ and put $\delta(\varepsilon) := \varepsilon/2\|f\|$. If $\delta(\varepsilon) \ge 1$, then there exists an element $y_\varepsilon \in X \setminus \mathrm{Ker}(f)$ such that

$$|(y, y_\varepsilon)_s| \le \delta(\varepsilon)\|y\| \, \|y_\varepsilon\| \quad \text{for all } y \in \mathrm{Ker}(f). \tag{8}$$

If $0 < \delta(\varepsilon) < 1$ and since the functional f is of (APP)-type there exists an element $y_\varepsilon \in X \setminus \mathrm{Ker}(f)$ such that (8) holds.

Put $z_\varepsilon := y_\varepsilon/\|y_\varepsilon\|$. Then for all $x \in X$ we have $y := f(x)z_\varepsilon - f(z_\varepsilon)x$ belongs to $\mathrm{Ker}(f)$ which implies, by (8), that:

$$\begin{aligned} &\left|\left(f(x)z_\varepsilon - f(z_\varepsilon)x, z_\varepsilon\right)_s\right| \le \delta(\varepsilon)\|f(x)z_\varepsilon - f(z_\varepsilon)x\| \le \\ &\le 2\delta(\varepsilon)\|f\| \, \|x\| \le \varepsilon\|x\| \quad \text{for all } x \in X. \end{aligned}$$

On the other hand, as above, we have:

$$\left(f(x)z_\varepsilon - f(z_\varepsilon)x, z_\varepsilon\right)_s = f(x) - \left(x, f(z_\varepsilon)z_\varepsilon\right)_i \quad \text{for all } x \in X$$

and denoting $x_\varepsilon := f(z_\varepsilon)z_\varepsilon \neq 0$ we obtain:

$$|f(x) - (x, x_\varepsilon)_i| \le \varepsilon\|x\| \quad \text{for all } x \in X.$$

If we replace x by $-x$ in the above estimation we get:

$$|f(x) - (x, x_\varepsilon)_s| \le \varepsilon\|x\| \quad \text{for all } x \in X$$

and the proof is finished.

Remark 3. The relation (7) is equivalent to:

$$|f(x) - (x, x_\varepsilon)_p| \le \varepsilon \quad \text{for all } x \in \bar{B}(0,1). \tag{7'}$$

Definition 2. *The normed linear space X is said to be of $(FAPP)$-type if every nonzero continuous linear functional on it is of (APP)-type.*

Some examples of (FAPP)-spaces will be given in the following.

4. ε-BIRKHOFF ORTHOGONALITY IN NORMED SPACES

Let X be a normed linear space over the real or complex number field $\mathbb{K}$ The following definition is a generalization of Birkhoff's orthogonality in normed spaces.

Definition 3. *Let $\varepsilon \in [0,1)$. The element $x \in X$ is said to be ε-Birkhoff orthogonal over $y \in X$ if*

$$\|x + \lambda y\| \ge (1 - \varepsilon)\|x\| \quad \text{for all } \lambda \in \mathbb{K}.$$

We denote $x \perp y\,(\varepsilon\text{-}B)$.

If A is a nonempty subset of X, then by $A^{\perp}\,(\varepsilon\text{-}B)$ we denote the set of all elements which are ε-BIRKHOFF orthogonal over A, i.e.,

$$A^{\perp}\,(\varepsilon\text{-}B) := \{y \in X \mid y \perp x\,(\varepsilon\text{-}B) \text{ for all } x \in A.\}$$

We remark that $0 \in A^{\perp}(\varepsilon\text{-}B)$ and $A \cap A^{\perp}\,(\varepsilon\text{-}B) \subseteq \{0\}$ for every $\varepsilon \in [0,1)$.

The following lemma is a variant of F. RIESZ result (see for example [**5**, p. 84]):

Lemma 1. *Let X be a normed space and G be its closed linear subspace. Suppose $G \neq X$. Then for any $\varepsilon \in (0,1)$ the ε-BIRKHOFF orthogonal complement of G is nonzero.*

Let $\bar{y} \in X \setminus G$. Since G is closed, $d(\bar{y}, E) = d > 0$. Thus there exists $y_\varepsilon \in G$ such that $d \leq \|\bar{y} - y_\varepsilon\| \leq d/(1-\varepsilon)$. Putting $x_\varepsilon := \bar{y} - y_\varepsilon$ we have $x_\varepsilon \neq 0$ and for all $y \in G$ and $\lambda \in \mathbb{K}$ we obtain:

$$\|x_\varepsilon + \lambda y\| = \|\bar{y} - y_\varepsilon + \lambda y\| = \|\bar{y} - (y_\varepsilon - \lambda y)\| \geq d \geq (1-\varepsilon)\|x_\varepsilon\|$$

what means that $x_\varepsilon \in G^{\perp}\,(\varepsilon\text{-}B)$. This completes the proof.

Note that the next decomposition theorem holds.

Theorem 5. *Let X be a normed linear space and G be its closed linear subspace. Then for any $\varepsilon \in (0,1)$ we have the decomposition: $X = G + G^{\perp}\,(\varepsilon\text{-}B)$.*

Suppose $G \neq X$ and $x \in X$. If $x \in G$, then $x = x + 0$ with $x \in G$ and $0 \in G^{\perp}\,(\varepsilon\text{-}B)$. If $x \notin G$, then there exists an element $y_\varepsilon \in G$ such that:

$$0 < d = d(x, G) \leq \|x - y_\varepsilon\| \leq \frac{d}{1-\varepsilon}.$$

Since $x_\varepsilon := x - y_\varepsilon \in G^{\perp}\,(\varepsilon\text{-}B)$ (see the proof of the above lemma) we conclude that $x = y_\varepsilon + x_\varepsilon$ with $y_\varepsilon \in G$ and $x_\varepsilon \in G^{\perp}\,(\varepsilon\text{-}B)$. This completes the proof.

5. ε-ORTHOGONALITY IN THE SENSE OF NORM DERIVATIVES

We shall begin with a definition.

Definition 4. *Let $\varepsilon \in [0,1)$. The element $x \in X$ (X is a real normed space) is called ε-orthogonal in the sense of semi-inner product $(\ ,\ )_p$ ($p = s$ or $p = i$) over the element $y \in X$ or x is p-orthogonal on y, for short, if*

$$|(y, x)_p| \leq \varepsilon \|x\|\,\|y\|. \tag{9}$$

We denote $x \perp y\,(\varepsilon\text{-}p)$.

If A is a nonempty subset of X then by $A^{\perp}\,(\varepsilon\text{-}p)$ we shall mean the set of all elements in X which are p-ε-orthogonal on A, i.e.,

$$A^{\perp}(\varepsilon\text{-}p) := \{y \in X \mid y \perp x\,(\varepsilon\text{-}p) \text{ for all } x \in A\}.$$

It is easy to see that $0 \in A^{\perp}\,(\varepsilon\text{-}p)$ and $A \cap A^{\perp}\,(\varepsilon\text{-}p) \subseteq \{0\}$ for all $\varepsilon \in [0,1)$ and if $A = -A$ then $A^{\perp}\,(\varepsilon\text{-}s) = A^{\perp}\,(\varepsilon\text{-}i)$ (and we denote $A^{\perp}(\varepsilon) := A^{\perp}(\varepsilon\text{-}i) = A^{\perp}(\varepsilon\text{-}s)$).

The next proposition is valid in the particular case of inner product spaces.

Proposition 1. *Let $(X;(\ ,\ ))$ be a real inner product space and $\varepsilon \in [0,1)$. Then the following statements hold:*

i) $x \perp y\,(\varepsilon\text{-}B)$ *iff* $x \perp y\,(\delta(\varepsilon))$ *where* $\delta(\varepsilon) := \sqrt{(2-\varepsilon)\varepsilon}$;

ii) $x \perp y\,(\varepsilon)$ *iff* $x \perp y\,(\eta(\varepsilon)\text{-}B)$ *where* $\eta(\varepsilon) := 1-\sqrt{1-\varepsilon^2}$.

i) It is clear that $x \perp y\,(\varepsilon\text{-}B)$ if and only if $\|x+ty\|^2 \geq (1-\varepsilon)^2\|x\|^2$, which is equivalent to: $\|y\|^2t^2 + 2(x,y)t - \varepsilon(\varepsilon-2)\|x\|^2 \geq 0$ for all $t \in \mathbb{R}$, i.e., $|(x,y)|^2 \leq \varepsilon(2-\varepsilon)\|x\|^2\|y\|^2$ and the statement is proved.

ii) Follows from i).

In virtue of this fact we can introduce the following concept.

Definition 5. *A real (smooth) normed space is called of (pSAPP)-type ((SAPP)-type) if there exists a mapping $\eta : [0,1) \to [0,1)$ such that:*

i) $\eta(\varepsilon)$ *iff* $\varepsilon = 0$;

ii) $x \perp y\,(\eta(\varepsilon)\text{-}B)$ *implies* $x \perp y\,(\varepsilon\text{-}p)$ *for all* $\varepsilon \in (0,1)$, *where* $p = s$ *or* $p = i$ $(p = s = i)$.

Remark 4. The previous proposition shows that every inner product space is a smooth normed space of (SAPP)-type. In Section 6 we shall point out other classes of smooth normed spaces of (SAPP)-type.

Lemma 2. *Let X be a normed space of (pSAPP)-type ($p = s$ or $p = i$). If G is a closed linear subspace in X and $G \neq X$, then for any $\varepsilon \in (0,1)$ the p-ε-orthogonal complement of G is nonzero and $G^{\perp}(\varepsilon\text{-}i) = G^{\perp}(\varepsilon\text{-}s)$ (we denote $G^{\perp}(\varepsilon)$).*

The proof is obvious from Lemma 1 observing that $G^{\perp}(\varepsilon\text{-}B) \subseteq G^{\perp}(\varepsilon\text{-}p)$ for all $\varepsilon \in (0,1)$ and to the fact that $G = -G$.

Using Theorem 5 we also have:

Theorem 6. *Let X be a normed space of (pSAPP)-type and G be its closed linear subspace. Then for every $\varepsilon \in (0,1)$ we have the decomposition*

$$X = G + G^{\perp}(\varepsilon).$$

Finally, we note that the following theorem holds.

Theorem 7. *If X is a real normed space of (pSAPP)-type ($p = s$ or $p = i$) then X is a real normed space of (FAPP)-type.*

Let f be a nonzero continuous linear functional on x and $\varepsilon \in (0,1)$. Then $G := \mathrm{Ker}(f)$ is a closed linear subspace in X and $G \neq X$. Applying Lemma 2 it follows that $G^{\perp}(\varepsilon\text{-}p)$ is nonzero, i.e., there exists an element $x_\varepsilon \in X \setminus \{0\}$ such that

$$|(y, x_\varepsilon)_p| \leq \varepsilon\|y\|\,\|x_\varepsilon\| \qquad \text{for all } y \in \mathrm{Ker}(f),$$

i.e., f is a functional of (APP)-type. This completes the proof.

Further on, we shall give some examples of normed spaces of (SAPP)-type which are not usual inner product spaces.

6. EXAMPLES OF NORMED SPACES OF (SAPP)-TYPE

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. If we denote $L^p(\Omega) \equiv L^p(\Omega, \mathcal{A}, \mu)$, $p > 1$, the real Banach space of p-integrable functions on Ω, then $L^p(\Omega)$ is smooth and:

$$\lim_{t\to 0} \frac{\|x+ty\|_p - \|x\|_p}{t} = \|x\|_p^{1-p} \int_\Omega x(s)^{p-1} \operatorname{sgn}(x(s))y(s)\,\mathrm{d}\mu(s)$$

for all $x, y \in L^p(\Omega)$, $x \neq 0$, (see for example [**1**, p. 314]).

Suppose $p \geq 2$ and put $p = 2k+2$. Then:

$$(y,x)_q = \|x\|_p^{-2k} \int_\Omega x^{2k+1}(s)y(s)\,\mathrm{d}\mu(s) \qquad (\text{here } q = s = i)$$

for all $x, y \in L^p(\Omega)$, $x \neq 0$, and $(y,0)_q = 0$ if $y \in L^p(\Omega)$.

If we put:

$$(y,x)'_q := \lim_{t\to 0} \frac{(y, x+ty)_q - (y,x)_q}{t}$$

then $(y,x)'_q$ exists for all $x, y \in L^p(\Omega)$ and a simple calculation shows that:

$$(10) \quad (y,x)'_q = (2k+1)\|x\|_p^{-2k} \int_\Omega x^{2k}(s)y^2(s)\,\mathrm{d}\mu(s) - 2k\|x\|_p^{-2k-2}\left[\int_\Omega x^{2k+1}(s)y(s)\,\mathrm{d}\mu(s)\right]^2.$$

On the other hand, by the Hölder inequality, we have:

$$\int_\Omega x^{2k}(s)y^2(s)\,\mathrm{d}\mu(s) \leq \left[\int_\Omega x^{2k+2}(s)\,\mathrm{d}\mu(s)\right]^{\frac{2k}{2k+2}} \left[\int_\Omega y^{2k+2}(s)\,\mathrm{d}\mu(s)\right]^{\frac{2}{2k+2}}$$

and

$$\left[\int_\Omega x^{2k+1}(s)\,\mathrm{d}\mu(s)\right]^2 \leq \left[\int_\Omega x^{2k+2}(s)\,\mathrm{d}\mu(s)\right]^{\frac{4k+2}{2k+2}} \left[\int_\Omega y^{2k+2}(s)\,\mathrm{d}\mu(s)\right]^{\frac{2}{2k+2}}.$$

Then from (10) we obtain the evaluation

$$(11) \qquad (y,x)'_q \leq (4k+1)\|y\|_p^2 \qquad \text{for all } \ x, y \in L^p(\Omega).$$

Proposition 2. *The* Banach *space $L^p(\Omega)$ with $p \geq 2$ is a smooth normed space of (SAPP)-type.*

Let consider the mapping $\varphi_{x,y} : \mathbb{R} \to \mathbb{R}$, $\varphi_{x,y}(t) := \|x+ty\|^2$ where x, y are given in $L^p(\Omega)$. Then $\varphi_{x,y}$ is two times differentiable on $\mathbb{R}$, the second derivative is nonnegative on $\mathbb{R}$ and

$$\varphi'_{x,y}(t) = 2(y, x+ty)_q, \qquad \varphi''_{x,y}(t) = 2(y, x+ty)'_q$$

for all $t \in \mathbb{R}$. Applying Taylor's formula for $\varphi_{x,y}$ we have

$$\|x+ty\|_p^2 = \|x\|_p^2 + 2t(y,x)_q + t^2(y, x+\xi_t y)'_q,$$

where $t \in \mathbb{R}$, ξ_t is between 0 and t and x, y are in $L^p(\Omega)$.

Using the inequality (11) we have, for all x, y in $L^p(\Omega)$ and t in $\mathbb{R}$

$$\|x+ty\|_p^2 - \|x\|_p^2 \le 2t(y,x)_q + (4k+1)t^2\|y\|_p^2.$$

It is clear that if $x \perp y\,(\varepsilon\text{-}B)$ then:

$$(\varepsilon^2 - 2\varepsilon)\|x\|_p^2 \le 2t(y,x)_q + (4k+1)t^2\|y\|_p^2 \qquad \text{for} \ \ t \in \mathbb{R},$$

which implies that $x \perp y\,(\gamma(\varepsilon))$ where $\gamma(\varepsilon) := \sqrt{\varepsilon(2-\varepsilon)(4k+1)}$. Putting $\lambda(\varepsilon) := 1 - \sqrt{1-\varepsilon^2/(4k+1)}$, $\varepsilon \in [0,1)$, we have $\lambda : [0,1) \to [0,1)$, $\lambda(\varepsilon) = 0$ iff $\varepsilon = 0$ and $x \perp y\,(\lambda(\varepsilon)\text{-}B)$ implies that $x \perp y\,(\varepsilon\text{-}q)$ for all $\varepsilon \in (0,1)$ $(q = s = i)$, i.e. $L^p(\Omega)$ is a smooth normed space of (SAPP)-type.

Corollary 1. *Let X_p be a linear subspace in $L^p(\Omega)$, $p \ge 2$, and G be its closed linear subspace. Then for all $\varepsilon \in (0,1)$ we have the decomposition $X_p = G + G^{\perp}(\varepsilon)$, where $G^{\perp}(\varepsilon)$ is taken in X_p.*

Corollary 2. *Let X_p be as above and f be a nonzero continuous linear functional on it. Then for all $\varepsilon \in (0,1)$, there exists a nonzero element x_ε in X_p so that:*

$$\left| f(x) - \|x_\varepsilon\|_p^{-2k} \int_\Omega x(s) x_\varepsilon^{2k+1}(s)\,\mathrm{d}\mu(s) \right| \le \varepsilon \left[\int_\Omega |x(s)|^p\,\mathrm{d}\mu(s) \right]^{\frac{1}{p}}.$$

REFERENCES

1. : *Variational Methods and Applications.* (Romanian), Ed. Acad. Bucureşti, 1980.
2. : *A characterization of best approximation elements in real normed linear spaces.* Studia Univ. Babeş-Bolyai, **33** (3) (1988), 74–80.
3. : *Characterization of reflexivity.* Studia Math., **23** (1964), 205–216.
4. : *A characterization of inner-product spaces.* Proc. Amer. Math. Soc., **41** (1973), 569–574.
5. : *Functional Analysis.* Springer-Verlag, Berlin, 1966.

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