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SOME INEQUALITIES FOR THE CHI SQUARE DISTRIBUTION FUNCTION

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We prove several inequalities for the incomplete Gamma function and the Chi square distribution function. A conjecture due to Ramanujan and proved by Karamata is generalized and some results regarding the function $\alpha \mapsto \mathrm{Prob}\,(\chi^2_\alpha \le \alpha)$ are obtained.

0. INTRODUCTION

In this paper we consider the following functions:

- *Incomplete Gamma function*

$$\gamma(u,v) = \int_0^v e^{-t} t^{u-1}\,\mathrm{d}t \qquad (u,v>0), \tag{1}$$

- *Chi square distribution function*

$$P(u,v) = \frac{\gamma(u,v)}{\Gamma(u)} = \mathrm{Prob}\,\big(\chi^2_{2u} \le 2v\big), \tag{2}$$

where χ^2_{2u} is a Chi square random variable with $2u$ degrees of freedom, and

- *the Gamma function*

$$\Gamma(u) = \int_0^\infty e^{-t} t^{u-1}\,\mathrm{d}t = \gamma(u,+\infty). \tag{3}$$

Note that for integer values of u all three functions can be expressed in terms of elementary functions. In particular, for $u=n$ we have

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$$P(n,v) = 1 - e^{-v}\left(1 + v + \frac{v^2}{2} + \frac{v^3}{3!} + \cdots + \frac{v^{n-1}}{(n-1)!}\right) \tag{4}$$

$$= \operatorname{Prob}(X > n-1),$$

where X is a Poisson random variable with the expectation equal to v.

Approximations for the function $P(u,v)$ are usually given in terms of various normal approximations (see [**1**] or [**2**]), valid as $v \to \infty$. A number of approximations and inequalities for functions γ and P, is given in the book [**3**].

Using logarithmic convexity of the Gamma function, we were able to produce (in [**4**]), new sharp bounds for the ratio $\Gamma(x+\beta)/\Gamma(x)$ and, in [**5**], bounds for $\Gamma(x)\Gamma(y)\big/\Gamma^2\big((x+y)/2\big)$. We now extend this method to a similar ratio with the function P. In the second part of the article, we show some properties of the function $x \mapsto P(x,x)$, thus generalizing an old conjecture due to Ramanujan and proved by Karamata in [**6**]. This result is related to the median of the Chi square distribution function. We also prove an inequality for a ratio $\gamma(x+\beta, x+\beta)/\gamma(x,x)$, and, as a consequence, we show that the function $P(x,x)$ is decreasing in x.

Let us note that some results obtained for the Chi square distribution function can be easilly extended to a more general case of the Gamma distribution function

$$P(u,v,\lambda) = \frac{\lambda^u}{\Gamma(u)} \int_0^v e^{-\lambda t} t^{u-1}\, \mathrm{d}t = \lambda^u P(u, \lambda v).$$

1. LOGARITHMIC CONVEXITY

It is easy to prove that the function $u \mapsto \gamma(u,v)$ is logaritmically convex on the domain $u > 0$, for any $v > 0$ (see, for instance, the classical book [**8**] by E. Artin). We proved in [**4**] that the function

$$G(u) = \log\Gamma(u) - \left(u - \frac{1}{2}\right)\log u - \frac{1}{12u} \tag{5}$$

is concave on $u > 0$. Therefore, the function

$$\varphi(u) = \log\gamma(u,v) - \log\Gamma(u) + \left(u - \frac{1}{2}\right)\log u + \frac{1}{12u} \tag{6}$$

is convex on $u > 0$, for any $v > 0$. Using Jenssen's inequality

$$\varphi(u+\beta) \le (1-\beta)\varphi(u) + \beta\varphi(u+1), \tag{7}$$

where $\beta \in [0,1]$, we can prove the following result:

Theorem 1. *For every $u > 0$, $v > 0$, $\beta \in [0,1]$ we have*

$$(8)\qquad \frac{P(u+\beta,v)}{P^{1-\beta}(u,v)P^{\beta}(u+1,v)} \leq$$

$$\frac{u^{(u-\frac{1}{2})(1-\beta)}(u+1)^{(u+\frac{1}{2})\beta}}{(u+\beta)^{u+\beta-\frac{1}{2}}}\exp\left(\frac{1-\beta}{12u}+\frac{\beta}{12(u+1)}-\frac{1}{12(u+\beta)}\right),$$

with the equality for $\beta = 0$ and $\beta = 1$.

A numerical investigation (see the table) shows that the inequality (8) sharpens as u grows larger. Using a similar technique as in [**4**], one can put more terms in the exponential, but it would improve the inequality only slightly. The main source of the error is a rather high second u-derivative of the function $\log\gamma(u,v)$, which can not be reduced bacause we have neither explicit form nor bounds for it.

This inequality can be used for an approximation of $P(u+\beta,v)$ for integer values of u.

Table: Relative errors in (8)

u	v	β	relative error (%)
2.00	2.00	0.10	1.38
2.00	2.00	0.30	3.07
2.00	2.00	0.50	3.47
2.00	2.00	0.70	2.76
2.00	2.00	0.90	1.12
4.00	2.00	0.10	0.39
4.00	2.00	0.30	0.87
4.00	2.00	0.50	1.00
4.00	2.00	0.70	0.82
4.00	2.00	0.90	0.34
12.00	10.00	0.10	0.13
12.00	10.00	0.30	0.29
12.00	10.00	0.50	0.34
12.00	10.00	0.70	0.28
12.00	10.00	0.90	0.12
24.00	19.00	0.10	0.05
24.00	19.00	0.30	0.11
24.00	19.00	0.50	0.13
24.00	19.00	0.70	0.10
24.00	19.00	0.90	0.04

2. FUNCTION $P(x,x)$

In this part we will investigate the function $x \mapsto P(x,x)$, for positive values of x.

Theorem 2. *For every* $x > 0$:

$$\frac{1}{2} + \frac{1}{3}\frac{x^x e^{-x}}{\Gamma(x+1)} < P(x,x) < \frac{1}{2} + \frac{1}{2}\frac{x^x e^{-x}}{\Gamma(x+1)}. \tag{9}$$

Proof. By an appropriate change of variables and an integration by parts in (1), it can be shown that

$$P(u,v) = \frac{v^u}{u}\left(e^{-v} + v\int_0^1 e^{-tv}t^u \,\mathrm{d}t\right).$$

If we define $\theta(x)$ by

$$P(x,x) = \frac{1}{2} + \theta(x)\frac{x^x}{\Gamma(x+1)e^x},$$

then we have

$$\theta(x) = 1 + \frac{x}{2}e^x\left(\int_0^1 \left(te^{-t}\right)^x \mathrm{d}t - \int_1^\infty \left(te^{-t}\right)^x \mathrm{d}t\right). \tag{10}$$

It is shown in [**6**] that θ of the form as in (10) decreases from $\frac{1}{2}$ to $\frac{1}{3}$ as x increases from 0 to $+\infty$; so the theorem is proved. ¤

As Karamata writes in [**6**], Ramanujan posed a problem [**7**]: Show that, if x is a positive integer,

$$\frac{1}{2}e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^{x-1}}{(x-1)!} + \frac{x^x}{x!}\theta(x),$$

where θ lies between $\frac{1}{2}$ and $\frac{1}{3}$. In the light of (4), this problem is a particular case of our Theorem 2.

Any of various normal approximations to the χ^2 distribution function leads to a conjecture that the median of χ^2_α has to be between $\alpha - 1$ and α, for large α. Using Theorem 2, we can prove more than that.

Corollary. *Let* M_α *be the median of* χ^2_α *random variable* $(\alpha > 1)$. *Then* $\alpha - 1 < M_\alpha < \alpha$.

Proof. The distribution function for χ^2_α is $P\left(\frac{\alpha}{2}, \frac{x}{2}\right)$. By Theorem 2, $P\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ is greater than $\frac{1}{2}$, and therefore $M_\alpha < \alpha$. The other half of the assertion can be proved by noticing that for $\alpha > 1$,

$$P\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)-P\left(\frac{\alpha}{2},\frac{\alpha-1}{2}\right)=\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int_{\frac{\alpha-1}{2}}^{\frac{\alpha}{2}} e^{-t}t^{\frac{\alpha}{2}-1}\,\mathrm{d}t>\frac{1}{2\Gamma\left(\frac{\alpha}{2}\right)}e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}-1},$$

where we used the fact that the function under the integral sign is decreasing within the limits of integration. Further, by (9),

$$\begin{aligned}(11)\qquad P\left(\frac{\alpha}{2},\frac{\alpha-1}{2}\right)&<P\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)-\frac{1}{2\Gamma\left(\frac{\alpha}{2}\right)}e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}-1}\\&=P\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)-\frac{1}{2\Gamma\left(\frac{\alpha}{2}+1\right)}e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}}\\&<\frac{1}{2}.\quad ¤\end{aligned}$$

Theorem 3.

(*i*) *For every $x>0$, $\beta>0$,*

$$(12)\qquad \frac{\gamma(x+\beta,x+\beta)}{\gamma(x,x)}<\frac{(x+\beta)^{x+\beta-1}}{x^{x-1}e^{\beta}}.$$

(*ii*) *The function $x\mapsto P(x,x)$ is decreasing in $x>0$.*

Proof. (*ii*) is obviously equivalent to

$$(13)\qquad \frac{\gamma(x+\beta,x+\beta)}{\gamma(x,x)}<\frac{\Gamma(x+\beta)}{\Gamma(x)},\qquad (x>0,\ \beta>0).$$

Kečkić and Vasić proved in [**9**]:

$$(14)\qquad \frac{\Gamma(x+\beta)}{\Gamma(x)}>\frac{(x+\beta)^{x+\beta-1}}{x^{x-1}e^{\beta}}.$$

So, (*ii*) will follow after we prove (*i*). After an appropriate change of variables in the integral, (12) becomes

$$(15)\qquad \int_0^x e^{-\frac{x+\beta}{x}t}t^{x+\beta-1}\,\mathrm{d}t<\frac{x^{\beta}}{e^{\beta}}\int_0^x e^{-t}t^{x-1}\,\mathrm{d}t.$$

By a mean value theorem,

$$(16)\qquad \int_0^x e^{-\frac{x+\beta}{x}t}t^{x+\beta-1}\,\mathrm{d}t=c^{\beta}e^{-\frac{\beta}{x}c}\int_0^x e^{-t}t^{x-1}\,\mathrm{d}t,\qquad c\in(0,x).$$

The function $c\mapsto c^{\beta}e^{-\frac{\beta}{x}c}$ reaches its maximum $x^{\beta}e^{-\beta}$ at $c=x$; therefore, (16) implies (15). ¤

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