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ON SOME CLASSES OF LINEAR EQUATIONS, VI

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We consider the equation (1) where P is a polynomial and A an algebraic linear operator on a linear space. The general solution of (1) is obtained under the conditions specified in Theorem 1 or Theorem 2. The theory of generalized inverses, when applied to the equation (1), does not lead to a better result.

1. Suppose that V is a linear space over C and that $A: V \to V$ is a linear operator. Once again we consider the equation in $x \in V$:

$$P(A)x = 0 \tag{1}$$

where P is a given complex polynomial.

In previous notes [**1**]–[**3**] we constructed the general solution of (1) assuming that V is a commutative algebra and by introducing some other restrictions on the operator A, mainly regarding the action of A on the product uv $(u, v \in V)$.

In this note we shall construct the general solution of (1), provided that A is an *algebraic operator* (i.e. provided that there exists an annihilating polynomial for A). In other words, we suppose that there exists the minimal polynomial M of A, for which we have, of course,

$$M(A) = 0. \tag{2}$$

It is then sufficient to consider the equation (1) where $\operatorname{dg} P < \operatorname{dg} M = m$.

2. We shall determine the general solution of (1) provided that the zeros of P and M have some special properties which will be specified later.

Namely, let D be the greatest common divisor of the polynomials P and M, and let

$$M(t) = D(t)F(t), \qquad P(t) = D(t)G(t). \tag{3}$$

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Then

$$x = F(A)u \qquad (u \in V \text{ is arbitrary}) \tag{4}$$

clearly satisfies the equation (1), since from (4) follows

$$P(A)x = P(A)F(A)u,$$

and in virtue of (3):

$$P(A)x = D(A)G(A)F(A)u = G(A)D(A)F(A)u = G(A)M(A)u = 0.$$

However, the solution (4), though it contains an arbitrary element $u \in V$, need not be the general solution of (1).

Example 1. Consider the matrix equation

$$Ax = 0, \tag{5}$$

where

$$A = \begin{Vmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad x = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix}.$$

In this case $M(t) = t^3 - 2t^2$, $P(t) = t$, $D(t) = t$, $F(t) = t^2 - 2t$, and

$$x = (A^2 - 2A)u \tag{6}$$

where u is an arbitrary 4×1 matrix is a solution of (5), but it is not its general solution. Indeed, the solution $x = \| 1 \;\; 2 \;\; -1 \;\; -1 \|^{\mathrm{T}}$ cannot be obtained from (6).

Hence, some other restriction is needed to ensure that (4) is the general solution of (1). In fact, it is enough to suppose that the polynomials F and P are relatively prime. Indeed, in that case there exist polynomials S and Q such that $F(t)S(t) + Q(t)P(t) = 1$, implying

$$F(A)S(A) + Q(A)P(A) = I. \tag{7}$$

Therefore, if x_0 is a solution of (1), i.e. if $P(A)x_0 = 0$, then from (7) follows: $F(A)S(A)x_0 + Q(A)P(A)x_0 = x_0$, i.e.

$$F(A)S(A)x_0 = x_0. \tag{8}$$

This means that if x_0 is a solution of (1), then there exists a polynomial S such that (8) is true. Hence, putting $u = S(A)x_0$ into (4), we find that (4) reduces to $x = x_0$; in other words, any solution x_0 of (1) is contained in the formula (4).

We have therefore proved the following theorem.

Theorem 1. *If D is the greatest common divisor of P and M, with $M(t) = D(t)F(t)$, and if F and P are relatively prime, then (4) is the general solution of (1).*

Remark. The conditions: F and P are relatively prime, and F and D are relatively prime, are clearly equivalent to each other.

Suppose now that $\lambda_1, \dots, \lambda_r$ are the only common (and distinct) zeros of P and M, and put

$$M(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_r)^{m_r} F(t), \quad P(t) = (t-\lambda_1)^{p_1} \cdots (t-\lambda_r)^{p_r} R(t), \tag{9}$$

where $F(\lambda_k) \neq 0$, $R(\lambda_k) \neq 0$ for $k = 1, \dots, r$. If

$$m_k \leq p_k \qquad (k = 1, \dots, r) \tag{10}$$

then

$$D(t) = (t-\lambda_1)^{m_1} \cdots (t-\lambda_r)^{m_r}$$

and the polynomials F and P (or equivalently F and D) are relatively prime. The converse is also true. Theorem 1 can therefore be reformulated in the following manner.

Theorem 2. *If $\lambda_1, \dots, \lambda_r$ are the only common and distinct zeros of P and M, such that (9) and (10) hold, then the general solution of (1) is given by (4).*

Corollary. *If all the zeros of M are simple, then the condition (10) is fulfilled.*

Example 2. Suppose that $M(t) = t^m - t^n$ ($m > n$ are nonnegative integers) is the minimal polynomial of A. If $n \geq 1$, the general solution of the equation $P(A)x = 0$ can be obtained provided that: either 0 is not a zero of P, or 0 is a zero of order $\geq n$ of P. In particular, if $n = 1$, then it is always possible to solve the equation $P(A)x = 0$.

Similarly, if $n = 0$, it is also always possible to solve the equation $P(A)x = 0$. As an illustration we mention the cyclic functional equation

$$a_{m-1}x(g^{m-1}t) + a_{m-2}x(g^{m-2}t) + \cdots + a_1 x(gt) + a_0 x(t) = 0, \tag{11}$$

where g maps a nonempty set S into itself, $g^m t = t$ ($t \in S$); $a_0, \dots, a_{m-1}$ are given complex numbers, and the unknown function x maps S into $\mathbf{C}$.

The equation (11) has the form (1) with $P(t) = a_{m-1}t^{m-1} + \cdots + a_0$, and $Ax(t) =: x(gt)$. Suppose that $t^m - 1 = D(t)F(t)$ where D is the g.c.d. of $t^m - 1$ and $P(t)$. Then the general solution of (11) is given by $x(t) = F(A)u(t)$, where $u: S \to \mathbf{C}$ is arbitrary.

This result is well known (see, for example [**4**], or [**5**] where it was proved in a manner similar to the proof of Theorem 1).

Remark. We have seen (Example 1) that (4) need not be the general solution of (1) if F and P are not relatively prime (or equivalently if (10) does not hold). On the other hand, those conditions are not necessary for the validity of the conclusions of Theorems 1 and 2, as shown by the following example.

Example 3. Consider the equation $Ax = 0$, where

$$A = \begin{Vmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 4 & -4 & -2 \end{Vmatrix}, \qquad x = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix}.$$

We then have $M(t) = t^3 - 2t^2$, $P(t) = t$, $D(t) = t$, $F(t) = t^2 - 2t$. The polynomials F and P are not relatively prime, but nevertheless the solution

$$x = F(A)u = (A^2 - 2A)u,$$

where u is an arbitrary 3×1 matrix is the general solution of the considered equation.

It should be noted that M and P from this example coincide with M and P from Example 1, but the respective conclusions do not.

3. It seems appropriate to investigate the equation (1) from the aspect of generalized inverses. Of course, the equation (1) is linear and its general solution is simply

$$x = (I - A_1^- A_1)u \qquad (u \in V \text{ arbitrary})$$

where A_1^- is a (1)-inverse of $A_1 \equiv P(A)$. This is certainly true, but useless. We shall therefore attempt to express the general solution of (1) in terms of a generalized inverse of A. It turns out that the Drazin inverse (although considered not to be an "equation solver"—see [**6**]) is the suitable inverse for this problem. We first briefly list a definition and some simple facts which will be needed later (for details consult [**6**]), and we then describe a method for solving (1).

If $M(t) = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_k t^k$ with $\alpha_k \neq 0$ is the minimal polynomial for A, we say that k is the *index* of A, and we write $k = \operatorname{Ind} A$.

Suppose that $k = \operatorname{Ind} A$. Then the following statements are true.

(*i*) *The equation $A^k x = 0$ is equivalent to the equation $A^n x = 0$ for all $n \geq k$.*

(*ii*) *If A^D is the* Drazin *inverse of A, then $(A^D)^k$ is a (1)-inverse of A^k.*

(*iii*) $A^D A = (A^D)^n A^n$, *where $n \in \mathrm{N}$.*

As a consequence of the above, we conclude that

(*iv*) *The general solution of the equation $A^k x = 0$ is $x = (I - A^D A)u$, where $u \in V$ is arbitrary.*

The following result, proved in [**7**], will also be used later.

(*v*) *If A and B are linear operators on V, such that $AB = BA$, $A^- B = BA^-$, where A^- denotes a (1)-inverse of A, then*

$$\ker AB = \ker A + \ker B.$$

Finally, we shall need the following fact:

(vi) *If M is the minimal polynomial of A, then the minimal polynomial of $A - \lambda I$ is:*

$$t^m + M^{(m-1)}(\lambda)t^{m-1} + \cdots + M'(\lambda)t + M(\lambda),$$

and also its consequence:

(vii) *For any nonnegative integer k the following two statements are equivalent:*

(a) $k = \operatorname{Ind}(A - \lambda I)$;

(b) *λ is a zero of order k of M.*

(We say that λ is a zero of order 0 of M if $M(\lambda) \neq 0$).

We now return to the equation (1), where A satisfies (2). Let $P(t) = (t-\lambda_1)^{p_1} \cdots (t - \lambda_n)^{p_n}$, so that (1) can be written in the form

$$(A - \lambda_1 I)^{p_1} \cdots (A - \lambda_n I)^{p_n} x = 0. \tag{12}$$

Let λ_k be a zero of order m_k of M, and suppose that

$$m_k \leq p_k \qquad (k = 1, \ldots, n). \tag{13}$$

Then using (vii) and (i) we see that (12) is equivalent to the equation

$$(A - \lambda_1 I)^{m_1} \cdots (A - \lambda_n I)^{m_n} x = 0. \tag{14}$$

Consider now the equation

$$(A - \lambda_k I)^{m_k} x = 0 \qquad (1 \leq k \leq n). \tag{15}$$

According to (iv) its general solution is

$$x = u - (A - \lambda_k I)^D (A - \lambda_k I) u,$$

where $u \in V$ is arbitrary. Hence, we can explicitely solve any equation of the form (15). But for all $k = 1, \ldots, n$ the operators $(A - \lambda_k I)^{m_k}$ commute, and also the operator $(A - \lambda_k I)^{m_k}$ commutes with a (1)-inverse of $(A - \lambda_j I)^{m_j}$ for all $k, j = 1, \ldots, n$. This follows from (ii) and the fact that A^D is a polynomial in A. Therefore, according to (v) we conclude that the general solution of (14) is the sum of the general solutions of the equations (15) for $k = 1, \ldots, n$.

In other words, it is possible to solve the equation (1) by this method provided that the condition (13) is fulfilled. The same result was also obtained in Section 2. However, the method of Section 2 also enables us to write down the general solution of (1) in a simple explicit form, and it is therefore preferable to the method of generalized inverses.

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