Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 2 (1991), 18–25.

A MEAN VALUE THEOREM FOR FOURIER COEFFICIENTS OF AN EVEN FUNCTION

Ilija Lazarević, Miodrag Ivović

In this paper functionals of a simple form are applied to Fourier series. In this way we obtain some analytical relations between Fourier coefficients and series sums in certain characteristic points of the convergence interval. Some properties of functions in question are characterized by devided differences. Such type of results could be of interest in the information theory.

Let $\mathcal{F}(f;x) = \frac{1}{2}a_0 + \sum\limits_{k=1}^{+\infty}(a_k \cos kx + b_k \sin kx)$ be the Fourier series of a function f in (a,b). In the present paper we shall consider an estimation of the coefficients a_k and b_k. From time to time such estimations appears in the literature in the form of an inequality which contains sums of the form

$$\sum_{i=1}^{m_0} c_{i0} f(x_{i0}) + \sum_{i=1}^{m_1} c_{i1} f'(x_{i1}) + \cdots + \sum_{i=1}^{m_p} c_{ip} f^{(p)}(x_{ip}), \tag{1}$$

where m_j are positive integers and where x_{ij} are knots from the segment $[a,b]$. In a case of nondifferentiable functions it is assumed that the terms containing derivatives are omitted. Majorant approximations of such type are not generally studied in mathematical literature and there are some particullar cases obtained by varions methods. In this paper we shall use the oportunity to introduce a general method which enables us to obtain aforementioned majorant approximations of Fourier coefficients. Basically we use the method of functionals of the simple form. Using this method it is possible to analyse all of the variants of such formulas. The same method makes it possible to analyse the influence of some function theoretic properties (such as the continuity, monotonicity, convexity and differentiability) on estimation of Fourier coefficients.

The definition of the functional of the simple form is dependent of the chosen Chebyshev system. We shall restrict ourselves here by chosing the Chebyshev

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1991 Mathematics Subject Classification: 46S30

system spanned on an ordinary algebraic polynomials system. The general theory of the functionals of the simple form is developed by T. Popoviciu in a large number of his papers (see, for example, [**1**], [**2**], [**3**]).

Definition 1. ([**2**]). *Let L be real, linear and bounded functional on $\mathbf{C}[a,b]$, $(-\infty < a < b < +\infty)$. For the functional L it is said that it is of the simple form if there is a constant K $(\neq 0)$ and there is $n \in \mathbf{N}$ (neither is dependent of f), and for every f there is an increasing sequence of knots $\xi_0, \xi_1, \dots, \xi_{n+1}$ such that for any $f \in \mathbf{C}[a,b]$*

$$L(f) = K[\xi_0, \xi_1, \dots, \xi_{n+1}; f], \tag{2}$$

where brackets denote usual divided differences of f.

In fact, the expression (2) is one of the representations of the functionals of the simple form. The above constants K and n, according that they are not dependent on f, are characteristic for L. The cardinal number of the above knots in the representation formulae (2) is called here "an index" of the same representation. If in (2) we take $f(x) = x^{n+1}$, we obtain $L(x^{n+1}) = K$ because of $\big[\xi_0, \xi_1, \dots, \xi_{n+1}; x^{n+1}\big] = 1$, which holds true for every sample of knots. In accordance to that it is possible to write

$$L(f) = L(x^{n+1})[\xi_0, \xi_1, \dots, \xi_{n+1}; f] \tag{3}$$

for any $f \in \mathbf{C}[a,b]$ instead of (2).

Definition 2. *For a linear functional $L\colon \mathbf{C}[a,b] \to \mathrm{R}$ we shall say that it is n-exact (i.e. that its degree of exactness is n) if $L(x^i) = 0$ for $i = 0, 1, \dots, n$ and $L(x^{n+1}) \neq 0$.*

Theorem 1. ([**2**]). *Bounded linear functional $L\colon \mathbf{C}[a,b] \to \mathrm{R}$ is of the simple form if and only if the following two conditions*

1° *L is n-exact,*

2° $L(x^{n+1})L_x\big((x-t)_+^n\big) \geq 0$

are satisfied for any $t \in [a,b]$, where

$$(x-t)_+^n = \begin{cases} 0, & \text{for } x < t \\ (x-t)^n, & \text{for } x \geq t \end{cases} \tag{4}$$

is well-known splyne-function (of the degree n) and where by L_x we have denoted that the functional L is applied to $(x-t)_+^n$ considered as a function of x.

Remark 1. If the domain of the functional L contains $(x-t)_+^0$ which is not continuous (its values by the definition are: 0 for $x < t$ and 1 for $x \geq t$), then the Theorem 1 is applicable to that case also.

Remark 2. The representation of the simple form can be considered also on some of the subspaces of $\mathbf{C}[a,b]$. In that case, if the space in consideration contains vectors $1, x, x^2, \dots, x^{n+1}$, then the criterion included in the above Theorem 1 can be used again.

From now on, we shall construct an example of majorization of the FOURIER coefficients on the basis of functionals of the simple form.

Let $f \in \mathbf{C}[-\pi,\pi]$ be an odd function. We suppose that f is differentiable in the points which are used. We shall consider approximation formulae of the following form

$$a_k \approx M \sum_{\nu=1}^{k} (-1)^{\nu-1} f'\left(\frac{2\nu-1}{2k}\pi\right), \tag{5}$$

where M is an constant which should be evaluated. For above formulas we are inspired by the properties of the function $\cos kx$. Namely the equidistancy of its zeros suggest the equidistancy of the knots of our approximation formulae and besides the convexity of $\cos kx$ implies the alternativity of weight-coefficients.

The following functional can be assigned to (5):

$$L(f) = a_k(f) - M \cdot s_k(f), \qquad s_k(f) = M \sum_{\nu=1}^{k} (-1)^{\nu-1} f'\left(\frac{2\nu-1}{2k}\pi\right), \tag{6}$$

where, regarding evenness of the function f, we have:

$$a_k(f) = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx \, \mathrm{d}x. \tag{7}$$

All knots $\frac{2\nu-1}{2k}\pi$ are in $[0,\pi]$, so we shall consider (6) on the linear space $\mathbf{C}[0,\pi]$ only.

In what follows we are going to use sums:

$$s_k^p = 1^p - 3^p + 5^p - \cdots + (-1)^{k-1}(2k-1)^p. \tag{8}$$

The following formulas can be derived

$$\begin{aligned} s_k^0 &= \frac{1-(-1)^k}{2}, \qquad s_k^1 = (-1)^{k-1}k, \\ s_k^2 &= \frac{(-1)^{k-1}(4k^2-1)-1}{2}, \qquad s_k^3 = (-1)^{k-1}k(4k-3). \end{aligned} \tag{9}$$

Further it is easy to show:

$$L(1) = a_k(1) - M \cdot 0 = 0,$$

$$L(x) = a_k(x) - M \cdot s_k^0 = \frac{2\left[(-1)^k - 1\right]}{\pi k^2} - M \frac{1-(-1)^k}{2},$$

$$L(x^2) = a_k(x^2) - M \frac{\pi}{k} s_k^1 = \frac{4(-1)^k}{k^2} - M\pi(-1)^{k-1},$$

$$
\begin{aligned}
L(x^3) &= a_k(x^3) - M\frac{3\pi^2}{4k^2}s_k^2 \\
&= -\frac{3\big[(-1)^k(4-2\pi^2k^2)-4\big]}{k^4} - M\frac{3\pi^2\big[(-1)^{k-1}(4k^2-1)-1\big]}{8k^2},
\end{aligned}
$$

$$
\begin{aligned}
L(x^4) &= a_k(x^4) - M\frac{\pi^3}{2k^3}s_k^3 \\
&= -\frac{(-1)^{k-1}8(6-\pi^2k^2)}{k^4} - M\frac{(-1)^{k-1}\pi^3(4k^2-3)}{2k^2}.
\end{aligned}
$$

One can immediately see that for k even, the functional L could have a larger degree of exactness. Thus, this case will be considered in the sequel.

Let us choose $M = -\frac{4}{\pi k^2}$ so that $L(x^2)$ becomes zero. It is easy to see that, in this case, $L(x^3) = 0$, and also $L(x^4) = -\frac{6\pi^2+48}{k^4} < 0$. Therefore, for even k we have an approximative formula

$$
a_k \approx -\frac{4}{\pi k^2}\sum_{\nu=1}^{k}(-1)^{\nu-1}f'\Big(\frac{2\nu-1}{2k}\pi\Big) \tag{10}
$$

of a high degree of exactness: $n = 3$. We shall prove, however, that the functional

$$
L(f) = a_k(f) + \frac{4}{\pi k^2}s_k(f), \tag{11}
$$

(which is adjoined to the above formula), is not of the simple form. To do this, let us calculate the value of this functional on the splain $(x-t)_+^3$. Using (4) we have

$$
a_k\big((x-t)_+^3\big) = \frac{2}{\pi}\int_t^{\pi}(x-t)^3\cos kx\,\mathrm{d}x = \frac{6}{\pi k^4}\big[k^2(\pi-t)^2 - 2(1-\cos kt)\big]. \tag{12}
$$

By $\frac{\mathrm{d}}{\mathrm{d}x}\big((x-t)_+^3\big) = 3(x-t)_+^2$, and using the relation (4), as well as relations (9), we obtain

$$
s_k\big((x-t)_+^3\big) = \begin{cases} 3\Big(-\frac{\pi^2}{2}+\pi t\Big), & \text{for } 0 \le t < \frac{\pi}{2k} \\ \sigma_k(\alpha;t), & \text{for } \frac{2\alpha-3}{2k}\pi \le t < \frac{2\alpha-1}{2k}\pi \\ 0, & \text{for } \frac{2k-1}{2k}\pi \le t \le \pi, \end{cases} \tag{13}
$$

where

$$
\begin{aligned}
\sigma_k(\alpha;t) &= 3\sum_{\nu=\alpha}^{k}(-1)^{\nu-1}\Big(\frac{2\nu-1}{2k}\pi - t\Big)^2 \\
&= -\frac{3\pi^2}{8k^2}\big[(-1)^{\alpha-2}(4\alpha^2-8\alpha+3)+4k^2-1\big] \\
&\quad + \frac{3\pi}{k}\big[(-1)^{\alpha-2}(\alpha-1)+k\big]t - \frac{3}{2}\big[1-(-1)^{\alpha-1}\big]t^2
\end{aligned} \tag{14}
$$

and α is the first index value in the above sum, taking values $\alpha = 2, 3, \dots, k$. By inspection of the expression on right hand side of (12) as a function of t, we conclude that $a_k\big((x-t)_+^3\big) > 0$ for $t \in (0, \pi)$. Considering this, as well as the third line in (13), we can derive inequality $L_x\big((x-t)_+^3\big) > 0$ for $t \in \big(\frac{2k-1}{2k}\pi, \pi\big)$. This fact, together with $L(x^4) < 0$ implies that the condition 2° in Theorem 1 does not hold and therefore, the functional L is not of a simple form.

In what follows we shall remain considering the case of even k, and we shall prove that the constant M in (6) can be determined in such a way that the functional L is of a simple form, but with a lower index. Let us first consider the sign of

$$L_x\big((x-t)_+^1\big) = a_k\big((x-t)_+^1\big) - M \cdot s_k\big((x-t)_+^1\big). \tag{15}$$

We find

$$a_k\big((x-t)_+^1\big) = \frac{2}{\pi}\int_t^{\pi}(x-t)\cos kx \,\mathrm{d}x = \frac{2}{\pi k^2}(1-\cos kt). \tag{16}$$

Denote by D the union of intervales $\big[\frac{2\alpha-3}{2k}\pi, \frac{2\alpha-1}{2k}\pi\big)$ for $\alpha = 2, 4, \dots, k-2$. Using (4) and (9) we have

$$\begin{aligned} s_k\big((x-t)_+^1\big) &= \sum_{\nu=1}^{k}(-1)^{\nu-1}(x-t)_+^0 \,\Big|_{\,x=\frac{2\nu-1}{2k}\pi} \\ &= \begin{cases} s_k^0 - s_\alpha^0 = -\frac{1-(-1)^{\alpha-1}}{2} = -1, & \text{for } \ t \in D \\ 0, & \text{for } \ t \in [0,\pi]\backslash D. \end{cases} \end{aligned} \tag{17}$$

Now it follows

$$L_x\big((x-t)_+^1\big) = \begin{cases} \frac{2}{\pi k^2}(1-\cos kt) + M, & \text{for } \ t \in D \\ \frac{2}{\pi k^2}(1-\cos kt), & \text{for } \ t \in [0,\pi]\backslash D. \end{cases} \tag{18}$$

From the last expression we conclude that following inequalities hold

$$\begin{aligned} \frac{2}{\pi k^2} + M \le L_x\big((x-t)_+^1\big) &\le \frac{4}{\pi k^2} + M \quad \text{for} \quad t \in D \\ 0 \le L_x\big((x-t)_+^1\big) &\le \frac{2}{\pi k^2} \quad \text{for} \quad t \in [0,\pi]\backslash D, \end{aligned} \tag{19}$$

so, for $M \ge -\frac{2}{\pi k^2}$ we have

$$L_x\big((x-t)_+^1\big) \ge 0 \quad \text{for} \quad t \in [0,\pi]. \tag{20}$$

Note that there is no M such that L_x is negative on spline $(x-t)_+^1$. For $M = -\frac{2}{\pi k^2}$ the functional

$$L(f) = a_k(f) + \frac{2}{\pi k^2}s_k(f) \tag{21}$$

will satisfy $L(1) = L(x) = 0$, $L(x^2) = \frac{2}{k^2}$ (> 0) and finally

$$L(x^2)L_x\big((x-t)_+^1\big) \geq 0 \quad \text{for} \quad t \in [0,\pi]. \tag{22}$$

From Theorem 1 it follows that the functional in (21) is of a simple form, which was our aim to prove.

So far we have been studying the problem of existence of a simple form functional. In what follows we shall show an application to an approximation of Fourier coefficients, based on the functional we studied in the first part of this paper.

Let us state again that the conclusion of simple form of the functional in (21) was derived under minimal set of assumptions; namely that the functional is defined on those functions in $\mathbf{C}[0,\pi]$ that are differentiable at points $\frac{2\nu-1}{2k}\pi$. Under such assumptions the following representation holds:

$$L(f) = \frac{2}{k^2}[\xi_0,\xi_1,\xi_2;f], \tag{23}$$

where ξ_0, ξ_1, ξ_2 are different points in $[0,\pi]$. If we put more assumptions on domain of L, we shall, of course, get more particular results. Let us state some of them.

1° Let f be a convex function on $[0,\pi]$. Then we know that $[\xi_0,\xi_1,\xi_2;f] \geq 0$ for every choice of different knots. Therefore, from (21) and (23) it follows

$$a_k(f) \geq -\frac{2}{\pi k^2}\sum_{\nu=1}^{k}(-1)^{\nu-1}f'\left(\frac{2\nu-1}{2k}\pi\right), \tag{24}$$

for a_k being cosine coefficients of an even function. The inequality (24) is a generalisation of a result in [**4**], obtained under more severe assumptions. Notice that, by previous considerations, it is obvious that the constant on right hand side of (24) is the best possible one. For a concave function f, clearly, the opposite inequality holds.

2° Formula

$$a_k(f) \approx -\frac{2}{\pi k^2}\sum_{\nu=1}^{k}(-1)^{\nu-1}f'\left(\frac{2\nu-1}{2k}\pi\right) \tag{25}$$

is a quadraturae formula (for $f(x)\cos kx$) and its residum is just the functional L. There are some known results for the estimation of best bounds for the divided differences (see for example [**5**]). Using the quotient representation of L one can obtain best bounds for the residum of the above quadratic formula.

3° Let us now assume that f has "better" properties. Namely, let $f \in \mathbf{C}^{(n+1)}[a,b]$, $(n \geq 0)$. Then there is a point η in the minimal interval containing all knots ξ_i such that

$$[\xi_0,\xi_1,\ldots,\xi_{n+1};f] = \frac{f^{(n+1)}(\eta)}{(n+1)!}, \tag{26}$$

[CAUCHY's formula]. Thus, a linear functional of a simple form admits the following integral representation too:

$$L(f)=\frac{f^{(n+1)}(\eta)}{(n+1)!} \tag{27}$$

for any $f\in \mathbf{C}^{(n+1)}[a,b]$.

In our particular case, we have for the functional in (2):

$$L(f)=\frac{f''(\eta)}{2} \tag{28}$$

for any $f\in \mathbf{C}^{(2)}[0,\pi]$. So, if a function possesses a corresponding differential property, we can use differentiability in analyzing the above approximation of FOURIER coefficients. More details on this see in [**7**], [**8**].

4° Finally, assume that all functions this procedure is applied to, have an integrable $(n+1)$-st derivative. Then for a linear functional of a simple form the following integral (PEANO's) representation holds (see [**9**]):

$$L(f)=\frac{1}{n!}\int_a^b f^{(n+1)}(t)\,L_x\big((x-t)_+^n\big)\,\mathrm{d}t \tag{29}$$

for any $f\in \mathbf{C}^{(n+1)}[a,b]$.

In our case we have thus:

$$a_k(f)=-\frac{2}{\pi k^2}\sum_{\nu=1}^{k}(-1)^{\nu-1}f'\left(\frac{2\nu-1}{2k}\pi\right)+\int_a^b f''(t)\,L_x\big((x-t)_+^1\big)\,\mathrm{d}t \tag{30}$$

for any $f\in \mathbf{C}^{(2)}[0,\pi]$.

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Faculty of Electrical Engineering,
University of Belgrade,
P. O. Box 816, 11001 Belgrade,
Yugoslavia

(Received March 9, 1990)

Faculty of Economics,
University of Belgrade,
Kamenička 6,
11000 Belgrade,
Yugoslavia