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NUMERICAL DIFFERENTIATION OF ANALYTIC FUNCTIONS

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Earlier D. Đ. Tošić obtained an infinite series representation for the central difference operator $\delta_\theta f(z) = f\left(z+\frac{1}{2}he^{i\theta}\right) - f\left(z-\frac{1}{2}he^{i\theta}\right)$, and used it to derive an n-point interpolation formula for the derivative $f'(z)$ of an analytic function $f(z)$. In this paper we give a direct proof of the n-point interpolation formula for $f'(z)$ with an error term $R_{1,n}$ expressed in integral form. Various other formulae are also included.

D. Đ. Tošić in [**1**] has shown that the operator

$$\delta_\theta f(z) = f\left(z+\frac{h}{2}e^{i\theta}\right) - f\left(z-\frac{h}{2}e^{i\theta}\right)$$

admits the representation

$$\delta_\theta = 2\sum_{m=1}^{\infty} \frac{1}{(2m-1)!}\left(\frac{hD}{2}\right)^{2m-1} e^{i(2m-1)\theta} \tag{1}$$

and this formula is used to obtain the numerical differentiation formula

$$Df(z) = \frac{1}{nh}\sum_{k=0}^{n-1} e^{-i\frac{k\pi}{n}}\left[f\left(z+\frac{h}{2}e^{i\frac{k\pi}{n}}\right) - f\left(z-\frac{h}{2}e^{i\frac{k\pi}{n}}\right)\right] + R_{1,n} \tag{2}$$

where the error term $R_{1,n}$ is given by the series

$$R_{1,n} = -\left(\frac{h}{2}\right)^{2n}\frac{D^{2n+1}f(z)}{(2n+1)!} - \left(\frac{h}{2}\right)^{4n}\frac{D^{4n+1}f(z)}{(4n+1)!} - \cdots .$$

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In this note a direct approach to the derivation of (1) and (2) is given with an integral form for the error term $R_{1,n}$ in (2), together with a finite series plus error term form for (1). Various other formulae are also derived. We begin with the algebraic identity

$$\frac{1}{t-z-\zeta} = \frac{1}{t-z} + \frac{\zeta}{(t-z)^2} + \cdots + \frac{\zeta^m}{(t-z)^{m+1}} + \frac{\zeta^{m+1}}{(t-z)^m(t-z-\zeta)},$$

and differentiation with respect to ζ gives

$$\frac{1}{(t-z-\zeta)^2} = \frac{1}{(t-z)^2} + \cdots + \frac{m\zeta^{m-1}}{(t-z)^{m+1}} + \frac{\zeta^m\big[(m+1)(t-z)-m\zeta\big]}{(t-z)^m(t-z-\zeta)^2}.$$

This, together with the CAUCHY integral formulae

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i}\oint_c \frac{f(t)}{(t-z)^{k+1}}\,\mathrm{d}t, \qquad (k=1,2,\ldots,m),$$

then gives us

$$f'(z+\zeta) = f'(z) + \zeta f''(z) + \cdots + \frac{\zeta^{m-1}f^{(m)}(z)}{(m-1)!} + \frac{1}{2\pi i}\oint_c \frac{f(t)\zeta^m\big[(m+1)(t-z)-m\zeta\big]}{(t-z)^m(t-z-\zeta)^2}\,\mathrm{d}t, \tag{3}$$

where c is a circle centre z and is such that $\zeta+z$ lies inside c. Using

$$\delta_\theta f(z) = \int_{-\frac{h}{2}e^{i\theta}}^{\frac{h}{2}e^{i\theta}} f'(z+\zeta)\,\mathrm{d}\zeta$$

and (3) with $m=2r-1$, we deduce the finite series plus error term form for (1), namely

$$\delta_\theta f(z) = 2\sum_{\nu=1}^{r}\frac{1}{(2\nu-1)!}\left(\frac{he^{i\theta}}{2}\right)^{2\nu-1} f^{(2\nu-1)}(z) + R_{2r-1}(\theta,h,z), \tag{4}$$

where

$$R_{2r-1}(\theta,h,z) = \frac{1}{2\pi i}\int_{-\frac{h}{2}e^{i\theta}}^{\frac{h}{2}e^{i\theta}}\oint_c \frac{f(t)\zeta^{2r-1}\big[2r(t-z)-(2r-1)\zeta\big]}{(t-z)^{2r-1}(t-z-\zeta)^2}\,\mathrm{d}t\,\mathrm{d}\zeta$$

and c is a circle centre z and radius greater than $\frac{|h|}{2}$.

Note 1. *To obtain (1) we simply use the infinite series*

$$\frac{1}{(t-z-\zeta)^2} = \frac{1}{(t-z)^2} + \frac{2\zeta}{(t-z)^3} + \cdots + \frac{m\zeta^{m-1}}{(t-z)^{m+1}} + \cdots$$

and integrate term-by-term as above.

With $r = 1$ in (4) we have

$$e^{-i\theta}\delta_\theta f(z) = hf'(z) + \frac{e^{-i\theta}}{2\pi i}\int_{-\frac{h}{2}e^{i\theta}}^{\frac{h}{2}e^{i\theta}} \oint_c \frac{f(t)\zeta\left[2(t-z)-\zeta\right]}{(t-z)(t-z-\zeta)^2}\,\mathrm{d}t\,\mathrm{d}\zeta,$$

and setting $\theta = \theta_k = \frac{k\pi}{n}$ gives, on summing over $k = 0, 1, \ldots, n-1$,

$$\sum_{k=0}^{n-1} e^{-i\theta_k}\delta_{\theta_k} f(z) = nhf'(z) + \sum_{k=0}^{n-1}\frac{e^{-i\theta_k}}{2\pi i}\int_{-\frac{h}{2}e^{i\theta_k}}^{\frac{h}{2}e^{i\theta_k}} \oint_c \frac{f(t)\zeta\left[2(t-z)-\zeta\right]}{(t-z)(t-z-\zeta)^2}\,\mathrm{d}t\,\mathrm{d}\zeta,$$

so that

$$Df(z) = \frac{1}{nh}\sum_{k=0}^{n-1} e^{-i\theta_k}\delta_{\theta_k} f(z) - \frac{1}{nh}\sum_{k=0}^{n-1}\frac{e^{-i\theta_k}}{2\pi i}\int_{-\frac{h}{2}e^{i\theta_k}}^{\frac{h}{2}e^{i\theta_k}} \oint_c \frac{f(t)\zeta\left[2(t-z)-\zeta\right]}{(t-z)(t-z-\zeta)^2}\,\mathrm{d}t\,\mathrm{d}\zeta,$$

which is (2) with a new form for the remainder $R_{1,n}$.

For higher derivatives of odd order we have

$$D^{2r-1}f(z) = \frac{2^{2r-2}(2r-1)!}{nh^{2r-1}}\sum_{k=0}^{n-1} e^{-i(2r-1)\theta_k}\delta_{\theta_k} f(z) + R_{2r-1,n}, \tag{5}$$

where the error term $R_{2r-1,n}$ is given in integral form. To obtain (5) we simply set $\theta = \theta_k$ in (4), multiply through by $e^{-i(2r-1)\theta_k}$ and sum over $k = 0, 1, \ldots, n-1$; the error term $R_{2r-1,n}$ then takes the form

$$R_{2r-1,n} = -\frac{2^{2r-2}(2r-1)!}{nh^{2r-1}}\sum_{k=0}^{n-1} e^{-i(2r-1)\theta_k} R_{2r-1}(\theta_k, h, z).$$

Next, we turn to the operator

$$\mu_\theta f(z) = \frac{1}{2}\left[f\left(z + \frac{h}{2}e^{i\theta}\right) + f\left(z - \frac{h}{2}e^{i\theta}\right)\right]$$

which, in [**1**], is shown to admit the representation

(6) $$\mu_\theta = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\frac{hD}{2}\right)^{2m} e^{i2m\theta},$$

and is used to obtain the numerical differentiation formula

(7) $$D^{2r} f(z) = \frac{(2r)!\, 2^{2r}}{nh^{2r}} \sum_{k=0}^{n-1} e^{-i2r\theta_k} \mu_{\theta_k} f(z) + R_{2r,n},$$

where the error term $R_{2r,n}$ is given by the series

$$R_{2r,n} = -(2r)! \sum_{\nu=1}^{\infty} \frac{1}{(2r+2\nu n)!} \left(\frac{h}{2}\right)^{2\nu n} D^{2r+2\nu n} f(z)$$

for $r = 1, 2, \ldots, n-1$. In case $r = n$, the corresponding formulae are

(8) $$D^{2n} f(z) = \frac{(2n)!\, 2^{2n}}{h^{2n}} \left[\frac{1}{n} \sum_{k=0}^{n-1} \mu_{\theta_k} f(z) - f(z)\right] + R_{2n,n}$$

and

$$R_{2n,n} = -(2n)! \sum_{\nu=1}^{\infty} \frac{1}{\big(2n(1+\nu)\big)!} \left(\frac{h}{2}\right)^{2\nu n} D^{2n(1+\nu)} f(z).$$

We now give a finite series plus error term form for (6) and indicate how (6) may be derived directly. In addition, (7) is derived with an integral form for the error term $R_{2r,n}$ $(r = 1, 2, \ldots, n)$ and the case $r = n$ is dealt with in similar fashion. We begin with the identity

$$\frac{t-z}{(t-z)^2 - \zeta^2} = \frac{1}{(t-z)} + \frac{\zeta^2}{(t-z)^3} + \cdots + \frac{\zeta^{2\nu}}{(t-z)^{2\nu+1}} + \frac{\zeta^{2\nu+2}}{(t-z)^{2\nu+1}\big[(t-z)^2 - \zeta^2\big]}.$$

This, together with

$$\begin{aligned} \mu_\theta f(z) &= \frac{1}{4\pi i} \oint_c f(t) \left(\frac{1}{t - z - \frac{h}{2}e^{i\theta}} + \frac{1}{t - z + \frac{h}{2}e^{i\theta}}\right) \mathrm{d}t \\ &= \frac{1}{2\pi i} \oint_c \frac{f(t)(t-z)}{(t-z)^2 - \frac{h^2}{4}e^{i2\theta}}\, \mathrm{d}t \end{aligned}$$

and $\zeta = \frac{h}{2}e^{i\theta}$ gives, on using the Cauchy integral formulae,

(9) $$\mu_\theta f(z) = \sum_{m=0}^{\nu} \left(\frac{he^{i\theta}}{2}\right)^{2m} \frac{f^{(2m)}(z)}{(2m)!} + R_{2\nu}(\theta, h, z)$$

which is (6) in finite series plus error term form, where the error term is given by

$$R_{2\nu}(\theta, h, z) = \frac{1}{2\pi i} \oint_c \frac{1}{(t-z)^{2\nu+1}\left[(t-z)^2 - \frac{h^2}{4}e^{i2\theta}\right]} \left(\frac{he^{i\theta}}{2}\right)^{2\nu+2} f(t)\,\mathrm{d}t$$

and c is a circle centre z and radius greater than $\frac{|h|}{2}$.

Note 2. *To obtain the representation formula (6) we use the infinite series*

$$\frac{t-z}{(t-z)^2 - \zeta^2} = \frac{1}{(t-z)} + \frac{\zeta^2}{(t-z)^3} + \cdots + \frac{\zeta^{2\nu}}{(t-z)^{2\nu+1}} + \cdots,$$

and integrate term-by-term as above.

Setting $\theta = \theta_k = \frac{k\pi}{n}$ and $\nu = r$ in (9), and multiplying through (9) by $e^{-i2r\theta_k}$ $(r = 1, 2, \ldots, n-1)$ and summing over $k = 0, 1, 2, \ldots, n-1$ $(n > 1)$ gives

$$\sum_{k=0}^{n-1} e^{-i2r\theta_k} \mu_{\theta_k} f(z) = n\left(\frac{h}{2}\right)^{2r} \frac{f^{(2r)}(z)}{(2r)!} + \sum_{k=0}^{n-1} R_{2r}(\theta_k, h, z) e^{-i2r\theta_k}$$

and so

$$D^{2r} f(z) = \frac{(2r)!\,2^{2r}}{nh^{2r}} \sum_{k=0}^{n-1} e^{-i2r\theta_k} \mu_{\theta_k} f(z) + R_{2r,n},$$

where

$$R_{2r,n} = -\frac{(2r)!}{n} \sum_{k=0}^{n-1} \frac{h^2 e^{i2\theta_k}}{8\pi i} \oint_c \frac{f(t)}{(t-z)^{2r+1}\left[(t-z)^2 - \frac{h^2}{4}e^{i2\theta_k}\right]}\,\mathrm{d}t.$$

In case $r = n$ we have

$$\sum_{k=0}^{n-1} \mu_{\theta_k} f(z) = nf(z) + n\left(\frac{h}{2}\right)^{2n} \frac{f^{(2n)}(z)}{(2n)!} + \sum_{k=0}^{n-1} R_{2n}(\theta_k, h, z)$$

and so

$$D^{2n} f(z) = \frac{(2n)!\,2^{2n}}{h^{2n}} \left[\frac{1}{n} \sum_{k=0}^{n-1} \mu_{\theta_k} f(z) - f(z)\right] - \frac{(2n)!\,2^{2n}}{nh^{2n}} \sum_{k=0}^{n-1} R_{2n}(\theta_k, h, z)$$

giving

$$D^{2n} f(z) = \frac{(2n)!\,2^{2n}}{h^{2n}} \left[\frac{1}{n} \sum_{k=0}^{n-1} \mu_{\theta_k} f(z) - f(z)\right] + R_{2n,n},$$

where

$$R_{2n,n} = -\frac{(2n)!}{n} \sum_{k=0}^{n-1} \frac{h^2 e^{i2\theta_k}}{8\pi i} \oint_c \frac{f(t)}{(t-z)^{2n+1}\left[(t-z)^2 - \frac{h^2}{4}e^{i2\theta_k}\right]}\,\mathrm{d}t$$

which is (8) with a new form for the error term $R_{2n,n}$.

In conclusion, we prove that

$$\sum_{k=0}^{n-1} e^{-i2\theta_k}\delta_{\theta_k}^2 = 2n\left(\frac{h^2D^2}{2!} + \frac{h^{2n+2}D^{2n+2}}{(2n+2)!} + \cdots + \frac{h^{2np+2}D^{2np+2}}{(2np+2)!}\right) + R_{2np+2,n}, \tag{10}$$

and indicate how the formula

$$\sum_{k=0}^{n-1} e^{-i2\theta_k}\delta_{\theta_k}^2 = 2n\left(\frac{h^2D^2}{2!} + \frac{h^{2n+2}D^{2n+2}}{(2n+2)!} + \frac{h^{4n+2}D^{4n+2}}{(4n+2)!} + \cdots\right) \tag{11}$$

may be established.

Now

$$\begin{aligned}\delta_\theta^2 f(z) &= f\left(z + he^{i\theta}\right) + f\left(z - he^{i\theta}\right) - 2f(z) \\ &= \int_0^{he^{i\theta}} \left(f'(z+\zeta) - f'(z-\zeta)\right) \mathrm{d}\zeta\end{aligned}$$

and using (3) with $2m$ in place of m we have

$$\begin{aligned}\delta_\theta^2 f(z) = &\int_0^{he^{i\theta}} \left(2\zeta f''(z) + \frac{2\zeta^3 f^{(4)}(z)}{3!} + \cdots + \frac{2\zeta^{2m-1} f^{(2m)}(z)}{(2m-1)!}\right) \mathrm{d}\zeta \\ &+ \frac{1}{2\pi i}\int_0^{he^{i\theta}} \oint_c \frac{f(t)\cdot 4\zeta^{2m+1}\left[(m+1)(t-z)^2 - m\zeta^2\right]}{(t-z)^{2m}\left[(t-z)^2-\zeta^2\right]^2} \,\mathrm{d}t\,\mathrm{d}\zeta \\ = &\, 2\left(\frac{h^2e^{i2\theta}D^2f(z)}{2!} + \frac{h^4e^{i4\theta}D^4f(z)}{4!} + \cdots + \frac{h^{2m}e^{i2m\theta}D^{2m}f(z)}{(2m)!}\right) \\ &+ R_{2m}(\theta, h, z).\end{aligned}$$

Setting $m = np+1$, $\theta = \theta_k = \frac{k\pi}{n}$ and summing over $k = 0, 1, \ldots, n-1$ after multiplying through by $e^{-i2\theta_k}$, we get

$$\begin{aligned}\sum_{k=0}^{n-1} e^{-i2\theta_k}\delta_{\theta_k}^2 f(z) = 2n\Bigg(&\frac{h^2D^2f(z)}{2!} + \frac{h^{2n+2}D^{2n+2}f(z)}{(2n+2)!} + \cdots \\ &+ \frac{h^{2np+2}D^{2np+2}f(z)}{(2np+2)!}\Bigg) + \sum_{k=0}^{n-1} e^{-i2\theta_k} R_{2np+2}(\theta_k, h, z)\end{aligned}$$

which is (10).

Note 3. *To obtain (11) we use the infinite series*

$$f'(z+\zeta) = f'(z) + \zeta f''(z) + \frac{\zeta^2}{2!} f'''(z) + \cdots$$

in place of (3) and proceed as above.

REFERENCES

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