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APPROXIMATION THEOREMS FOR FUNCTIONS CONVEX WITH RESPECT TO CHEBYSHEV SYSTEM

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The approximation of a function, which is convex with respect to a given Chebyshev system $\{u_0, u_1\}$ is considered. Two theorems are proved: The first one states the convexity of a sequence of linear combination of truncated "linear" functions in the sense of Chebyshev. The second theorem establishes the estimation of an error in approximation of a Chebyshev-convex function by such linear combinations. In a special case, if one takes the system $\{1, x\}$, the known results of Popoviciu and Toda are obtained.

0. A number of classical theorems of L. Galvani [**1**], K. Toda [**2**] and T. Popoviciu [**3**] consider the problem of approximation of a convex function by picewise affine function. In [**4**] some generalisations of this theorem were given. In this paper, the problem of approximation of function convex with respect to Chebyshev system $\{u_0, u_1\}$ is studied.

1. Let $-\infty < a < b < +\infty$ and $I = [a, b]$. For a continuous real function $f\colon I \to \mathrm{R}$ we shall say that it is convex on I with respect to Chebyshev system $\{u_0, u_1\}$ if

$$U\begin{pmatrix} u_0,\, u_1,\, f \\ x_1,\, x_2,\, x_3 \end{pmatrix} = \begin{vmatrix} u_0(x_1) & u_1(x_1) & f(x_1) \\ u_0(x_2) & u_1(x_2) & f(x_2) \\ u_0(x_3) & u_1(x_3) & f(x_3) \end{vmatrix} \ge 0, \tag{1}$$

whenever $a < x_1 < x_2 < x_3 < b$. In this case, according to [**1**], it will be said that f belongs to the cone $C\,(u_0, u_1)$, of functions convex with respect to $\{u_0, u_1\}$.

It is known ([**5**]) that the functions u_0 and u_1, forming the Chebyshev system on $[a, b]$, can be explicitelly expressed using functions w_0 and w_1, positive on I, such that $w_0 \in \mathbf{C}^1[a, b]$ and $w_1 \in \mathbf{C}[a, b]$. Namely, we have

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(2) $$u_0(x) = w_0(x), \qquad u_1(x) = w_0(x) \int_a^x w_1(t)\,\mathrm{d}t, \qquad a \le x \le b.$$

For a convex cone C we denote by $W = C \cap (-C)$ the maximal vector subspace contained in C (see [**6**]).

As $\pm u_0 \in C\,(u_0, u_1)$, $\pm u_1 \in C\,(u_0, u_1)$, we see that W is exactly the set $\{u_0, u_1\}$ of indepedant solutions of equation

(3) $$U\begin{pmatrix} u_0,\ u_1,\ f \\ x_1,\ x_2,\ x_3 \end{pmatrix} = 0.$$

2. In this section we shall introduce some assumptions and notations.

Let $\sigma_n(I)$ be a division of the interval $I = [a, b]$

(4) $$a = x_0 < x_1 < \cdots < x_n = b,$$

and

(5) $$I_k = \begin{cases} [x_k, x_{k+1}), & 0 \le k \le n-2, \\ [x_{n-1}, x_n], & k = n-1. \end{cases}$$

It is obvious that

$$I = \bigcup_{k=0}^{n-1} I_k = [a, b], \qquad I_k \cap I_j = \emptyset \qquad (k \ne j).$$

The characteristic function δ_k is defined, as usually, by

(6) $$\delta_k(x) = \begin{cases} 1, & x \in I_k \\ 0, & x \notin I_k \end{cases} \qquad (0 \le k \le n-1),$$

and the function $x \mapsto h_c(x) = \begin{cases} 1, & x \ge c \\ 0, & x < c \end{cases}$. Let

(7) $$h_k(x) \overset{\text{def}}{=} h_{x_k}(x).$$

Obviously, $h_0(x) \equiv 1$ on $[a, b]$, and $h_n(b) = 1$.

The following relationships between the functions (6) and (7)

(8) $$\delta_k(x) = h_k(x) - h_{k+1}(x) \qquad (0 \le k \le n-2),$$

(9) $$\delta_{n-1}(x) = h_{n-1}(x)$$

hold, for every $x \in I$. On the basis of (8) and (9), we have

$$h_m(x) = \sum_{k=m}^{n-1} \delta_k(x) \qquad (m = 0, 1, \dots, n-1). \tag{10}$$

3. Let the Chebyshev system $\{u_0, u_1\}$ be given by (2). The set of the solutions of inequality (1) forms the cone $C(u_0, u_1)$ with the maximal subspace W with the basis $\{u_0, u_1\}$. Moreover, we have

$$W = \operatorname{span}\left\{ u_0, u_1 \,\middle|\, u_0(x) = w_0(x);\ u_1(x) = w_0(x) \int_a^x w_1(t)\,\mathrm{d}t \right\}.$$

Let the functions $L_k \in W$ $(k = 0, 1, \dots, n-1)$ be given by

$$L_k(x) = A_k w_0(x) + B_k w_0(x) \int_{x_k}^x w_1(t)\,\mathrm{d}t \qquad (k = 0, 1, \dots, n-1), \tag{11}$$

(A_k, B_k are real constants). If functions $L_k(x)$ interpolate the function f in points x_k and x_{k+1}, then $L_k(x_k) = f(x_k)$, $L_k(x_{k+1}) = f(x_{k+1})$, from which we get

$$A_k = \frac{f(x_k)}{w_0(x_k)}, \qquad B_k = \frac{A_{k+1} - A_k}{J_k(x_{k+1})} \qquad (k = 0, 1, \dots, n-1), \tag{12}$$

where

$$J_k(x) = \int_{x_k}^x w_1(t)\,\mathrm{d}t \qquad (k = 0, 1, \dots, n-1). \tag{13}$$

Definition 1. *A generalized polygonal line of order n in the* Chebyshev *system $\{u_0, u_1\}$ is the function φ_n which is*

1° *continuous and bounded on I,*

2° *for arbitrary $\sigma_n(I)$, $\varphi_n(x) \equiv L_k(x)$, $x \in I_k$.*

Lemma 1. *Let $f \in \mathbf{C}[a, b]$ and σ_n of $[a, b]$ be given. Then, the generalized polygonal line of order n has a form*

$$\varphi_n(x) = \sum_{k=0}^{n-1} \delta_k(x) L_k(x) \qquad (a \le x \le b), \tag{14}$$

where δ_k is given by (6), L_k by (11).

Proof. It is easy to prove, by direct checking, that $\varphi_n(x)$, given by (14), satisfies 1° and 2° in Definition 1.

The function φ_n can be expressed in a more suitable form.

Lemma 2. *The alternative form for* φ_n *reads*

$$\varphi_n(x) = A_0 u_0(x) + B_0 u_1(x) + \sum_{k=1}^{n-1} m_k \varphi_1(x; x_k) \tag{15}$$

where the functions u_0 *and* u_1 *are given by (2),* A_0 *and* B_0 *by (12),* m_k *with*

$$m_k = B_k - B_{k-1} \qquad (k = 1, 2, \dots, n-1), \tag{16}$$

and where

$$\varphi_1(x; c) = \begin{cases} 0, & a \le x \le c, \\ w_0(x) \int\limits_c^x w_1(t)\,\mathrm{d}t, & c \le x \le b. \end{cases} \tag{17}$$

Proof. From (14), (8), (9) and (10), we obtain

$$\begin{aligned} \varphi_n(x) &= \sum_{k=0}^{n-2} \delta_k(x) L_k(x) + \delta_{n-1}(x) L_{n-1}(x) \\ &= \sum_{k=0}^{n-2} \big(h_k(x) - h_{k+1}(x)\big) L_k(x) + h_{n-1}(x) L_{n-1}(x), \end{aligned}$$

which, in virtue of (11) and (12), we can write, omitting arguments of the functions, in the form

$$\begin{aligned} \varphi_n(x) = -w_0 \sum_{k=0}^{n-2} A_k \Delta h_k + \sum_{k=0}^{n-2} B_k w_0 J_k h_k - \sum_{k=0}^{n-2} B_k w_0 J_k h_{k+1} \\ + A_{n-1} w_0 h_{n-1} + B_{n-1} w_0 J_{n-1} h_{n-1}, \end{aligned}$$

i.e.

$$\varphi_n(x) = -w_0 \sum_{k=0}^{n-2} A_k \Delta h_k + A_{n-1} w_0 h_{n-1} + \sum_{k=0}^{n-1} B_k w_0 J_k h_k - \sum_{k=0}^{n-2} B_k w_0 J_k h_{k+1}. \tag{18}$$

It immediately follows that

$$\varphi_1(x; x_k) = w_0(x) J_k(x) h_k(x),$$

where h_k, J_k and $\varphi_1(x; x_k)$ are given by (7), (13) and (17). If we introduce

$$D_k = J_k - J_{k+1}, \tag{19}$$

the equality (18) becomes

$$\varphi_n(x) = -w_0 \sum_{k=0}^{n-2} A_k \Delta h_k + A_{n-1} w_0 h_{n-1} - w_0 \sum_{k=0}^{n-2} B_k D_k h_{k+1} + \sum_{k=0}^{n-1} B_k \varphi_1(x; x_k) - \sum_{k=1}^{n-1} B_{k-1} \varphi_1(x; x_k),$$

i.e.

$$\varphi_n(x) = w_0(x)\alpha_n(x) + \sum_{k=1}^{n-1} m_k \varphi_1(x; x_k), \tag{20}$$

where we use (16) and introduce

$$\alpha_n(x) = B_0 J_0 h_0 + A_{n-1} h_{n-1} - \sum_{k=0}^{n-2} A_k \Delta h_k - \sum_{k=0}^{n-2} B_k D_k h_{k+1}. \tag{21}$$

Further, we have $h_0(x) = 1$ for $x \in [a, b]$. On the basis of (12) and (19) it immediately follows that

$$D_k \cdot B_k = A_{k+1} - A_k. \tag{22}$$

The relations (13), (21) and (22) give

$$\alpha_n(x) = B_0 J_0 + \sum_{k=0}^{n-2} A_k h_k + A_{n-1} h_{n-1} - \sum_{k=0}^{n-2} A_{k+1} h_{k+1} = B_0 J_0 + A_0. \tag{23}$$

Substituting (23) in (20) we get (15) which proves the lemma.

Lemma 3. *If $f \in C(u_0, u_1)$ on $[a, b]$, then the coeffitients m_k in (15) are nonnegative for every $k = 1, 2, \ldots, n-1$.*

Proof. From (12), (13) and (16) we have

$$m_k = \frac{A_{k+1} - A_k}{J_k(x_{k+1})} - \frac{A_k - A_{k-1}}{J_{k-1}(x_k)}. \tag{24}$$

On the other hand, division σ_n of $[a, b]$ fulfils the condition $x_{k-1} < x_k < x_{k+1}$, and according to the supposition of lemma that $f \in C(u_0, u_1)$, we get

$$U\begin{pmatrix} u_0,\ u_1,\ f \\ x_{k-1},\ x_k,\ x_{k+1} \end{pmatrix} = u_0(x_{k-1}) u_0(x_k) u_0(x_{k+1}) V(k) \geq 0 \tag{25}$$

where $V(k)$ denotes the determinant

$$\begin{vmatrix} 1 & J_0(x_{k-1}) & A_{k-1} \\ 1 & J_0(x_k) & A_k \\ 1 & J_0(x_{k+1}) & A_{k+1} \end{vmatrix}$$

which is nonnegative as a consequence of $u_0(x_{k-1})u_0(x_k)u_0(x_{k+1}) > 0$. After evident transformations we have

$$V(k) = (A_{k+1} - A_k)J_{k-1}(x_k) - (A_k - A_{k-1})J_k(x_{k+1}).$$

From (24), (25), and the above equality we find

$$m_k = \frac{V(k)}{J_{k-1}(x_k)J_k(x_{k+1})}, \qquad k = 1, 2, \ldots, n-1.$$

In virtue of positivity of $J_{k-1}(x_k)$ and $J_k(x_{k+1})$ and nonnegativity of $V(k)$ we obtain that $m_k \geq 0$ $(k = 1, 2, \ldots, n-1)$.

Lemma 4. *Let the division σ_n of the interval $[a, b]$, be given. Every function $x \mapsto \varphi_n(x)$ given by (15), where A_0 and B_0 are arbitrary real constants, and where $m_k \geq 0$ $(k = 1, 2, \ldots, n-1)$ belongs to the cone $C(u_0, u_1)$ on $[a, b]$.*

Proof. The structure of convex cone $C(u_0, u_1)$ ensures that $-u_0$ and $-u_1$ belong to it. Furthermore, for arbitrary A_0 and B_0, we have $A_0u_0 + B_0u_1 \in C(u_0, u_1)$. It is known ([**5**] p. 381) that every function $\varphi_1(x; x_k)$ belongs to $C(u_0, u_1)$ $(k = 1, 2, \ldots, n-1)$. Accordingly, every linear combination of functions $\varphi_1(x; x_k)$, with nonnegative coefficients m_k belongs to the cone $C(u_0, u_1)$. This completes the proof.

Now, we are going to consider the question of the uniform convergence of the sequence $\big(\varphi_1(x)\big)_1^\infty$ toward the function f. Namely, the following lemma takes place.

Lemma 5. *Let the function $f\colon [a, b] \to \mathrm{R}$ be continuous (from the right in $x = a$ and from the left in $x = b$). Let the interval $[a, b]$ be divided by equidistant σ_n division i.e. let*

$$x = a + \nu h, \qquad h = \frac{b-a}{n} \qquad (\nu = 0, 1, \ldots, n). \tag{26}$$

Then, the sequence of functions φ_n defined by (15), and where A_k, B_k, m_k and $\varphi_1(x; x_k)$ are given by (12), (16) and (17), converges uniformly toward the function f on $[a, b]$.

Proof. Let the function g be defined by

$$g(x) = \frac{f(x)}{w_0(x)}, \qquad x \in [a, b]. \tag{27}$$

This functions is continuous in virtue of continuity of f and positivity and continuity of w_0. From (11), (12) and (13) for every $x \in I_p = [x_p, x_{p+1}]$ we have

(28) $$L_p(x) = g(x_p)w_0(x) + \frac{g(x_{p+1}) - g(x_p)}{J_0(x_{p+1}) - J_0(x_p)} w_0(x)\big(J_0(x) - J_0(x_p)\big),$$

and

(29) $$\varphi_n(x) = L_p(x) \quad \text{for} \quad x \in I_p.$$

On the basis of (28) and (29) we get

(30) $$\begin{aligned} |f(x) - \varphi_n(x)| &= |f(x) - L_p(x)| \\ &= |w_0(x)| \left| g(x) - g(x_p) - \frac{g(x_{p+1}) - g(x_p)}{J_0(x_{p+1}) - J_0(x_p)} \big(J_0(x) - J_0(x_p)\big) \right| \\ &\le |w_0(x)| \left\{ \left| \frac{J_0(x_{p+1}) - J_0(x)}{J_0(x_{p+1}) - J_0(x_p)} \right| |g(x) - g(x_p)| \right. \\ &\quad \left. + \left| \frac{J_0(x) - J_0(x_p)}{J_0(x_{p+1}) - J_0(x_p)} \right| |g(x) - g(x_{p+1})| \right\}. \end{aligned}$$

As $w_0 \in \mathbf{C}[a, b]$, then there exists a constant $K > 0$ such that

(31) $$K = \max_{a \le x \le b} |w_0(x)|.$$

On the other hand, from the fact that $w_1(x) > 0$, we have

(32) $$0 < \frac{J_0(x_{p+1}) - J_0(x)}{J_0(x_{p+1}) - J_0(x_p)} = \left[\int_x^{x_{p+1}} w_1(t)\,\mathrm{d}t \right] \left[\int_{x_p}^{x_{p+1}} w_1(t)\,\mathrm{d}t \right]^{-1} < 1$$

and

(33) $$0 < \frac{J_0(x) - J_0(x_p)}{J_0(x_{p+1}) - J_0(x_p)} = \left[\int_{x_p}^{x} w_1(t)\,\mathrm{d}t \right] \left[\int_{x_p}^{x_{p+1}} w_1(t)\,\mathrm{d}t \right]^{-1} < 1$$

because of $x \in I_p$. On the basis of relation (30)–(33) we obtain

(34) $$\begin{aligned} |f(x) - \varphi_n(x)| &\le K \big(|g(x) - g(x_p)| + |g(x) - g(x_{p+1})|\big) \\ &\le 2K\omega_g(h) = 2K\omega_g\left(\frac{b-a}{n}\right), \end{aligned}$$

where ω_g is the modulus of continuity of the function g, defined by (27), and $\omega_g\big(\frac{b-a}{n}\big) \to 0$ when $n \to \infty$, as a consequence of continuity of g.

Remark 1. A proof of this lemma can be derived without the supposition of equidistantness of division σ_n. In this case it is sufficient to ensure that $\lim_{n\to\infty} \max_{1\le k\le n} (x_k - x_{k-1}) = 0$.

From the Lemmas 1–5, we obtain the following theorem:

Theorem 1. *Every function φ_n $(n = 1, 2, \dots)$ given by (15) where A_0 and B_0 are arbitrary real constants and $m_k \ge 0$ $(k = 1, 2, \dots, n-1)$ is convex with respect to the* Chebyshev *system $\{u_0, u_1\}$ on $[a, b]$, i.e. $\varphi_n \in C(u_0, u_1)$.*

Theorem 2. *Every function $f \in C(u_0, u_1)$ continuous on (a, b) and continuous from the right in a and from the left in b, is an uniform limit of the sequence of generalized polygonal lines $(\varphi_n)_0^\infty$ defined by (15), where A_0, $B_0 \in \mathrm{R}$ and $m_k \ge 0$ $(k = 1, 2, \dots, n-1)$.*

Remark 2. As it is explicitely mentioned in theorem, f must be continuous from the right in the point $x = a$ and from the left in $x = b$, since every function f which belongs to the cone $C(u_0, u_1)$ on $[a, b]$, has the property of continuity on the open interval (a, b), i.e. $f \in C(a, b)$ (see [**5**] p. 380). Thus, the suppositions for the end points are justified.

Remark 3. In some special cases, our theorems reduce to the known theorems. For example, for $u_0(x) = 1$, $u_1(x) = x$ we obtain the results apearing in the works of L. Galvani [**1**], K. Toda [**2**] and T. Popoviciu [**3**]. In the case $u_0(x) = \cos x$, $u_1(x) = \sin x$ or $u_0(x) = \cosh x$, $u_1(x) = \sinh x$ we obtain a theorem from [**4**].

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