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# ON POSITIVE DEFINITE FUNCTIONS DEFINED ON VECTOR SPACES

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**We consider positive definite functions on topological vector spaces and some of their properties.**

**0. Introduction.** In this paper we consider positive definite functions (PDF), defined on arbitrary linear topological vector spaces. Starting from the general definition of a PDF we first give a review of known results using only elementary methods in their proofs. Further we obtain a generalisation of a result known as HERGLOTZ lemma, and give some results for PDFs on normed vector spaces.

The theory of PDFs can be developed in a more general context of semigroups (see [**3**]). Inequalities for PDFs of the type that we consider in this paper are usually derived under weaker assumption that $f$ is a characteristic function of a probability measure, making proofs much more involved than the proofs that we suggest in the Theorem 4. These inequalities are useful in probability theory (see [**4**] and [**5**]). In the last several years there has been a considerable amount of work in probability theory on infinite dimensional spaces (see, for example [**6**] or [**7**]) and the results of the type we consider in this paper may be useful in this area.

**1. Definition.** *We say that a finite, complex–valued function $f$, defined on a linear vector space $E$ is positive definite, abbreviated PDF, if, for all finite $A = (a_1, \dots, a_n) \in \mathbf{C}^n$ and $x = (x_1, \dots, x_n) \in E^n$ the following inequality holds:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \bar{a}_j f(x_i - x_j) \geq 0. \tag{1}$$

**2. Lemma.** *If $f$ is a PDF then:*

$$f(0) \geq 0 \tag{2}$$

$$\bar{f}(x) = f(-x) \tag{3}$$

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***Proof.*** Inequality (2) follows from (1) for $A = 1$, $X = x$. To obtain (3), let us take $A = (-1, 1)$, $X = (0, x)$. Then (1) becomes:

$$2f(0) - f(x) - f(-x) \geq 0, \tag{4}$$

which, together with (2) gives $f(x) + f(-x) \in \mathbf{R}$, i.e., $f(x) + f(-x) = \bar{f}(x) + \bar{f}(-x)$, so we have

$$\Im f(x) = -\Im f(-x). \tag{5}$$

Now let $A = (1, i)$, $X = (0, x)$. From (1) it follows

$$2f(0) + \imath\,(f(x) - f(-x)) \geq 0. \tag{6}$$

By (3) we have $f(x) - f(-x) = \imath r$, $r \in \mathbf{R}$, thus $f(x) - f(-x) = -\bar{f}(x) + \bar{f}(-x)$, i.e., $\Re f(x) = \Re f(-x)$, which, together with (5) gives (3).

**3. Remark.** Several simple, but useful properties can be derived directly from (1): If $f$, $f_n$, $g$ are PDFs, then so are $\bar{f}$, $\Re f$, $af + bg$ for $a, b \geq 0$, and $\lim\limits_{n\to\infty} f_n$ if exists. It can be proved, using HILBERT space theory, that $f$ is a PDF if and only if there exists a family of functions $m_k \colon E \to \mathbf{C}$, $k \in K$, where $K$ is an index set, so that $\sum\limits_{k\in K} |m_k(x)|^2 < \infty$ for every $x \in E$ and

$$f(x - y) = \sum_{k\in K} m_k(x)\bar{m}_k(y),$$

for every $x, y \in E$. Using this fact, one can easily prove that if $f$ and $g$ are PDF then so is their product $fg$.

Only these elementary statements are enough to show that the following are examples of PDFs:

$(i)$ $|f|$ and $|f|^2$ if $f$ is a PDF.

$(ii)$ $g(f)$, where $f$ is a PDF and $g(u) = \sum\limits_{k=0}^{\infty} a_k u^k$, $a_k \geq 0$.

$(iii)$ $f(x) = \exp(\imath\phi(x))$, where $\phi$ is a real additive functional on $E$.

$(iv)$ Functions defined on $\mathbf{R}$: $e^{\imath ax}$, $(\cos ax)^k$, $(2 - \cos ax)^{-k}$, for $a \in \mathbf{R}$, $k \in \mathbf{N}$.

**4. Theorem.** *Let $f$ be a PDF. Then for every $x, y \in E$ we have:*

$(i)$ $|f(x)| \leq f(0)$.

$(ii)$ $|f(x) - f(y)|^2 \leq 2f(0)\,(f(0) - \Re f(x - y))$.

$(iii)$ $f(0) - \Re f(2x) \leq 4\,(f(0) - \Re f(x))$.

***Proof.*** By 3$(i)$, it is enough to prove $(i)$ for a real–valued $f$; in that case, $(i)$ is a consequence of (3) and (4). If $f(0) = 0$ then $f(x) = 0$ for all $x$ by $(i)$; so it is enough to prove $(ii)$ under assumption $f(0) = 1$. So, assume $f(0) = 1$ and let $X = (x, y, 0)$, $A = (-1, 1, a)$. Then (1) gives:

$$2\Re\,(\bar{a}\,((f(x) - f(y))) \leq 2\,(1 - \Re f(x - y)) + |a|^2. \tag{7}$$

Taking now $a = f(x) - f(y)$, one gets $(ii)$.

Inequality $(iii)$ also suffices to be proved in the case $f(0) = 1$ only. In this case $(iii)$ can be obtained from (1) with $X = (0, x, x, -x)$ and $A = (-1, -1, 1, 1)$.

**5. Remark.** Using inequality $(ii)$ of Theorem 4, one can see that a positive definite function $f$ defined on a linear topological vector space $E$ is uniformly continuous on $E$ it is continuous at 0 if and only if $\Re f$ is continuous at 0. Functions with this property are of a special interest in Probability theory; by Bochner's theorem, every such a function is the characteristic functional of a random variable.

**6. Lemma.** *If $f$ is a PDF and $|f(t)| = f(0)$ for some $t \in E$, $t \neq 0$, then there is an $a \in \mathbf{R}$ such that*

$$f(x+t) = f(0)e^{\imath a} f(x) = f(x)f(t), \tag{8}$$

*for every $x \in E$.*

***Proof.*** If $f(0) = 0$ then $f(x) = 0$ for all $x \in E$ and (11) holds for any $a$. So, assume for simplicity that $f(0) = 1$.

Let us first consider the case $f(t) = 1$ for some $t \in E$, $t \neq 0$. By Theorem 4$(ii)$ with $y = x + t$ we have $f(x) = f(x+t)$ for every $x \in E$. In the general case, if $|f(t)| = 1$, there is an $a \in \mathbf{R}$ so that $f(t) = e^{\imath a}$. Let $\phi$ be a real linear functional on $E$ such that $\phi(t) = -a$. Then the function $g(x) = f(x)e^{\imath\phi(x)}$ is a PDF, as a product of two PDFs, and $g(0) = g(t) = 1$. By the previous particular case, we have $g(x+t) = g(x)$, which gives (11). The case $0 < f(0) \neq 1$ results by noticing that $h = f/f(0)$ is a PDF and $h(0) = 1$.

**7. Lemma.** *If $f$ is a PDF and for all $x \in E$ we have $|f(x)| = f(0)$, then $f(x) = f(0)e^{\imath\phi(x)}$, where $\phi$ is a real additive functional on $E$, i.e., $\phi(x+y) = \phi(x) + \phi(y)$.*

***Proof.*** By assumption, $f(x) = f(0)e^{\phi(x)}$ for some real valued function $\phi$. Additivity of $\phi$ follows from Lemma 6.

**8. Remark.** As a corollary to Lemmas 6 and 7, we have the following result on characteristic functions of $E'$–valued random variables (i.e., continuous positive definite functions on $E$ with $f(0) = 1$), generalised here for an arbitrary linear topological space (see [**2**], p. 475).

**9. Remark.** Let $E$ be a linear topological vector space and $E'$ its topological dual. Let $f$ be the characteristic function of an $E'$–valued random variable. Then there exist only the following three possibilities:

$(i)$ $|f(x)| < 1$ for all $x \in E$, $x \neq 0$.

$(ii)$ $|f(t)| = 1$ for some $t \neq 0$. In this case $f(x+t) = f(x)f(t)$, and along every ray $L = \{x \in E : x = kx_0\}$ we have either $|f(x)| < 1$ for all $x \neq 0$ in a neighborhood of 0, or $f(x) = 1$ for all $x \in L$.

$(iii)$ $f(x) = 1$ for all $x \in E$; in this case $f(x) = e^{\imath\phi(x)}$ for some $\phi \in E'$.

***Proof.*** Follows by the continuity of $f$ and Lemmas 6 and 7.

**10. Example.** Let $E$ be a normed vector space, $\dim E \geq 2$, and let $S$ be a proper subspace of $E$. By HAHN–BANACH theorem there is a real linear functional $\phi \in E'$ so that $\phi(x) = 0$ for all $x \in S$ and $\phi$ is not identically zero. The characteristic function $f(x) = e^{\imath\phi(x)}$ is equal to one on $S$. This shows that in $(ii)$ above, the phrase "along every ray" cannot be omitted, even in the two–dimensional case.

**11. Lemma.** *Let $F$ be a function defined on $E$ and $Q \subset E$. Let $I_Q$ be the indicator function of $Q$. For $x \in E$ define a function $g$ by*

$$g(x) = f(x) I_Q(x), \tag{9}$$

*Then $f$ is a PDF if and only if $g$ is a PDF for all sets $Q \subset E$ satisfying*

$$0 \in Q \tag{10}$$

$$x \in Q, \quad y \in Q \Rightarrow x - y \in Q. \tag{11}$$

***Proof.*** Suppose that $f$ is a PDF. Let $A = (a_1, \dots, a_n)$, $X = (x_1, \dots, x_n)$, and define $S = \{1, \dots, n\}$, $J = \{(i,j) \in S^2 : x_i - x_j \in Q\}$. Define the relation $\rho$ on $S$ by $i\rho j \Leftrightarrow (i,j) \in J$. Using (10) and (11) one can easily show that $\rho$ is an equivalence; let $J_1, \dots, J_m$ be equivalence classes such that $S = \bigcup_{k=1}^{m} J_k$. Then we have:

$$\begin{aligned} \sum_{i,j=1}^{n} a_i \bar{a}_j g(x_i - x_j) &= \sum_{(i,j)\in J} a_i \bar{a}_j f(x_i - x_j) \\ &= \sum_{k=1}^{m} \sum_{(i,j)\in J_k} a_i \bar{a}_j f(x_i - x_j). \end{aligned}$$

Since $J_1, \dots, J_m$ are disjoint and $f$ is a PDF, we conclude that all inner sums above are non–negative and the assertion follows.

To show the converse, suppose that $g$ is a PDF for all sets $Q$ that satisfy (10) and (11). For $X = \{x_1, \dots, x_n\}$ let $Q$ be the set of all linear combinations of points from $X$. Then the function $g$ defined by (9) is a PDF by assumption, and

$$\sum_{i,j=1}^{n} a_i \bar{a}_j f(x_i - x_j) = \sum_{i,j=1}^{n} a_i \bar{a}_j g(x_i - x_j) \geq 0,$$

and, therefore, $f$ is a PDF.

**12. Example.** Let $Q = \{qc : k = 0, \pm 1, \pm 2, \dots\}$, $c =$ const. Then Lemma 12 gives that $f$ is a PDF if and only if each "sample function" of $f$ is a PDF. This result is known as HERGLOTZ lemma (see [**1**], p. 220).

**13. Theorem.** *Suppose that $E$ is a normed space and $f$ is a PDF defined on $E$, such that $\lim_{x\to\infty} f(x) = c$. Then $\Re c \geq 0$.*

***Proof.*** Assume $f(0) = 1$, with no loss of generality. Let $y_1, \dots, y_n, \dots$ be a sequence in $E$ such that $\|y_i - y_j\| \ge |i-j|$. Let $x_{i,n} = y_{i^n}$ for $i, n = 1, 2, \dots$ . Without difficulties it can be proved that the following holds:

$$\lim_{n\to\infty} \left| \sum_{\substack{i,j=1,2,\dots,n\\ i<j}} \frac{f(x_{i,n} - x_{j,n})}{n(n-1)} - \frac{c}{2} \right| = 0. \tag{12}$$

Let now $A = (1, 1, \dots, 1)$, $X = (x_{1,n}, x_{2,n}, \dots, x_{n,n})$. Then (1) reads:

$$n + 2\Re \sum_{\substack{i,j=1,2,\dots,n\\ i<j}} f(x_{i,n} - x_{j,n}) \ge 0. \tag{13}$$

Dividing (13) by $n(n-1)$, letting $n \to \infty$ and using (12), we obtain $\Re c \ge 0$.

**14. Theorem.** *Let $E$ be a normed space and suppose $f(x) = f(0) + g(x) + h(x)$, where $f(0) \ge 0$, $g(-x) = -g(x)$ and $g(x) = o(\|x\|)$, $h(x) = o(\|x\|^2)$ as $x \to 0$. Then $f$ is a PDF if and only if it is equal to a constant.*

***Proof.*** Assume all conditions above, and suppose that $f$ is a PDF. It is obviously sufficient to consider the case $f(0) = 1$. From the above assumptions it follows that $|f(x)|^2 = 1 + o(\|x\|^2)$. Therefore:

$$0 = \lim_{x\to 0} \frac{1 - |f(x)|^2}{\|x\|^2} = 2 \lim_{x\to 0} \frac{1 - |f(x)|}{\|x\|^2}.$$

By Theorem 4$(ii)$ we have

$$\left| \frac{|f(x+y)| - |f(y)|}{\|x\|} \right|^2 \le 2 \cdot \frac{(1 - f(x)}{\|x\|^2},$$

so the Fréchet derivative of $|f|$ is zero at every point $y \in E$, and $|f| = \text{const}$. By the continuity of $f$ and Lemma 7, we have that $f(x) = e^{\imath\phi(x)}$, where $\phi$ is a continuous linear functional on $E$. But then $f(x) = 1 + \imath\phi(x) + o(\|x\|)$, which is not of the assumed form unless $\phi(x) = 0$ for all $x \in E$, and so $f$ is a constant.

**15. Example.** Let $\phi$ be a continuous linear functional on $E$, and let $k \ge 2$. Then $f(x) = e^{-\imath\phi^k(x)}$ is not a PDF.

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