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744. ON THE APPROXIMATION OF THE LOGARITHMIC FUNCTION BY SEQUENCES OF ALGEBRAIC FUNCTIONS (I)*

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When the logarithmic function is approximated by sequences of algebraic functions, similar questions can be posed as in the case of other similar problems. So, for example, it is interesting to investigate the possibility of two-sided approximations in the same class of functions, the degree of the approximation, the speed of convergence of the approximation sequences, their monotony, and so on.

In this paper we construct a few sequences of algebraic functions by which we approximate the logarithmic function. We also point out some properties of those approximations.

I

Let us suppose that $\mathcal{A}:=\{A_n(x)\}$, $\mathcal{B}:=\{B_n(x)\}$, $\mathcal{C}:=\{C_n(x)\}$, $n=1, 2, \ldots$ are three sequences of functions, defined by

$$(1)\qquad \begin{aligned} &A_n(x):=\frac{2n(x^{1/n}-1)}{(x-1)(x^{1/n}+1)},\quad B_n(x):=\frac{n(x^{1/n}-1)}{(x-1)\,x^{1/2n}},\\ &C_n(x):=\frac{n(x^{1/n}-1)(x^{1/3n}+1)}{(x-1)(x^{1/n}+x^{1/3n})},\quad n=1, 2, \ldots;\ x>0,\ x\neq 1.\end{aligned}$$

Theorem 1. *The above sequence $\mathcal{A}$ is increasing and the sequences $\mathcal{B}$ and $\mathcal{C}$ are decreasing and at the some time we have*

$$(2)\qquad A_n(x)<\frac{\ln x}{x-1}<C_n(x)<B_n(x)\qquad n=1, 2, \ldots;\ x>0,\ x\neq 1.$$

Proof. For an arbitrary, but fixed $x\in(0, 1)\cup(1, +\infty)$ let us define the function $\varphi:=\varphi_x:(0, 1)\to \mathbf{R}$ by

$$(3)\qquad \varphi(y)=\varphi_x(y)=\frac{2(x^y-1)}{(x-1)\,y\,(x^y+1)}.$$

Then we have

$$(4)\qquad A_n(x)=\varphi\left(\frac{1}{n}\right),\qquad n=1, 2, \ldots$$

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For the function φ we find that

$$(5)\qquad \varphi'(y)=\frac{2}{x-1}\cdot\frac{2x^y y\ln x-(x^{2y}-1)}{y^2(x^y+1)^2}=\frac{2}{x-1}\cdot\frac{x^y(2y\ln x-(x^{2y}-1)x^{-y})}{y^2(x^y+1)^2}$$
$$=\frac{2x^y(\ln x^{2y}-(x^{2y}-1)x^{-y})}{y^2(x-1)(x^y+1)^2}.$$

Now we will use the following two known inequalities (see for example [1; p. 273])

$$(6)\qquad \frac{2}{t+1}<\frac{\ln t}{t-1}<\frac{1}{\sqrt{t}},\qquad t>0,\ t\neq 1.$$

Introducing the substitution $t=x^{2y}$ into the second of those inequalities, we obtain

$$(7)\qquad \frac{\ln x^{2y}-(x^{2y}-1)x^{-y}}{x^{2y}-1}<0,\qquad x>0,\ x\neq 1.$$

From this and from (5) we conclude that

$$(8)\qquad \varphi'(y)<0,\qquad y>0.$$

Further on, taking into account that $1/(n+1)<1/n$, we obtain

$$(9)\qquad \varphi\left(\frac{1}{n+1}\right)>\varphi\left(\frac{1}{n}\right),\qquad n=1,2,\dots,$$

wherefrom and from the above it follows that

$$(10)\qquad A_n(x)<A_{n+1}(x),\qquad n=1,2,\dots;\ x>0,\ x\neq 1,$$

holds true. In such a way we have obtained that the sequence $\mathcal{A}$ is increasing. Putting $t=x^{1/n}$ is the first of the inequalities (6) it follows that

$$(11)\qquad A_n(x)<\frac{\ln x}{x-1},\qquad n=1,2,\dots;\ x>0,\ x\neq 1.$$

In connection with the sequence $\mathcal{B}$ we can consider the function $\psi=\psi_x:(0,1/2)\to\mathbf{R}$ which is for every $x\in(0,+\infty)$ defined by

$$(12)\qquad \varphi(y):=\varphi_x(y):=\frac{x^{2y}-1}{x^y y}.$$

Now, we have

$$(13)\qquad B_n(x)=\frac{1}{2(x-1)}\psi\left(\frac{1}{2n}\right),\qquad n=1,2,\dots.$$

From (12) it follows that

$$(14)\qquad \varphi'(y)=\frac{x^{2y}+1}{x^y y^2}\left(\ln x^y-\frac{x^{2y}-1}{x^{2y}+1}\right),$$

Putting $t=x^{2y}$ into the first of the inequalities (6), we conclude that

$$(15)\qquad \frac{\ln x^y}{x^{2y}-1}>\frac{1}{x^{2y}+1},\qquad x>0,\ x\neq 1.$$

Introducing $1/x$ instead of x, it can be easily seen that if our inequalities are valid for $x \in (0, 1)$ then those inequalities are valid also for $x \in (1, +\infty)$. Hence, it is sufficient to prove our statements only for one of those intervals. From (14) and (15) it follows that

$$\psi'(y) > 0, \qquad x \in (1, +\infty). \tag{16}$$

By the use ot the above result and the equality (12), we obtain

$$\frac{1}{2(x-1)} \psi\left(\frac{1}{2n}\right) > \frac{1}{2(x-1)} \psi\left(\frac{1}{2n+2}\right), \qquad x > 0, \ x \neq 1, \tag{17}$$

i.e.

$$B_n(x) > B_{n+1}(x), \qquad n = 1, 2, \dots; \ x > 0, \ x \neq 1. \tag{18}$$

Hence, the sequence $\mathcal{B}$ is increasing. Putting $t = x^{1/n}$ in the second of the inequalities (6) we have

$$\frac{\ln x}{x-1} < \frac{1}{x^{1/2n}}, \tag{19}$$

which, on the basis on what we have said above, can be written in the form

$$\frac{\ln x}{x-1} < B_n(x), \qquad n = 1, 2, \dots; \ x > 0, \ x \neq 1. \tag{20}$$

The sequence $\mathcal{C}$ is more complicate than the other sequences but the approximation obtained by the use of that sequence is better than the approximation obtained by the use of the sequences $\mathcal{A}$ or $\mathcal{B}$.

Let us consider the function $f = f_x : (0, 1/3) \to \mathbf{R}$ defined by

$$f(y) = f_x(y) := \frac{(x^{3y} - 1)(x^y + 1)}{y(x^{3y} + x^y)} \tag{21}$$

for every, but fixed, $x \in (0, +\infty)$. Now, we have

$$C_n(x) = \frac{1}{3(x-1)} f\left(\frac{1}{3n}\right), \qquad n = 1, 2, \dots. \tag{22}$$

The function f can be transformed in the following way

$$f(y) = \frac{(x^{2y} - x^{-2y}) + (x^y - x^{-y})}{y(x^y + x^{-y})} = \frac{\operatorname{sh} 2t + \operatorname{sh} t}{y \operatorname{ch} t}\Big|_{t = y \ln x} \tag{23}$$

$$= y^{-1}(2 \operatorname{sh} t + \operatorname{th} t)\big|_{t = y \ln x},$$

wherefrom we have

$$f'(y) = y^{-2} \Phi(y), \tag{24}$$

where

$$\Phi(y) = -2 \operatorname{sh} t - \operatorname{th} t + (2 \operatorname{ch} t + \operatorname{ch}^{-2} t) t \big|_{t = y \ln x}. \tag{25}$$

Now, we shall examine the function Φ (for $x > 1$ and $y > 0$).

First, we find

$$\lim_{y \to +\infty} \Phi(y) = 2 \lim_{t \to +\infty} (t \operatorname{ch} t - \operatorname{sh} t) + \lim_{t \to +\infty} (t \operatorname{ch}^{-2} t - \operatorname{th} t) = +\infty. \tag{26}$$

Further we have

(27) $$\Phi'(y)=2t(1-\mathrm{ch}^{-3}t)\,\mathrm{sh}\,t\ln x\,|_{t=y\ln x}$$

wherefrom we conclude that

(28) $$\Phi'(y)<0, \qquad x>1,\ y>0,$$

and consequently

(29) $$\Phi(y)>\Phi(0)=0, \qquad x>1,\ y>0,$$

In virtue of (24) and (29) it follows that

(30) $$f'(y)>0, \qquad x>1,\ y>0.$$

Further on, since we have

(31) $$\lim_{y\to 0+} f(y)=\lim_{y\to 0+}\frac{(2\,\mathrm{sh}\,t+\mathrm{th}\,t)\,|_{t=y\ln x}}{y}=\lim_{t\to 0+}(2\,\mathrm{ch}\,t+\mathrm{ch}^{-2}t)\ln x=3\ln x,$$

we obtain

(32) $$f(y)>\lim_{y\to 0+} f(y)=3\ln x>0, \qquad x>1,\ y>0.$$

From what we have said about we can conclude that the function f is increasing for every, but fixed, $x\in(1,+\infty)$.

For $x\in(0,1)$, $y>0$, the sign in the inequalities from (28), (29), (30) and (32) are contrary, wherefrom we conclude that the function f is decreasing for every, but fixed, $x\in(0,1)$.

If, in addition, we define the function F by

(33) $$F(x,y):=\frac{f(y)}{3(x-1)}, \qquad x>0,\ x\neq 1;\ y>0,$$

then, on the basis of (32), we infer that

(34) $$F(x,y)>0,$$

as such as we conclude that the function F is increasing in variable $y(>0)$ for every fixed $x>0$, $x\neq 1$. Since the inequality $1/3(n+1)<1/3n$ holds for every $n\in\mathrm{N}$ we have

(35) $$C_{n+1}(x)=F\left(x,\frac{1}{3(n+1)}\right)<F\left(x,\frac{1}{3n}\right)=C_n(x), \quad n=1,2,\ldots;\ x>0,\ x\neq 1.$$

This proves that the sequence $\mathscr{C}$ is monotone.

To prove that the sequence approximates the function $\frac{\ln x}{x-1}$ from above, we will use one more well known inequality (see for example [1; p. 273]):

(36) $$\frac{\ln t}{t-1}<\frac{1+\sqrt[3]{t}}{t+\sqrt[3]{t}}, \qquad t>0,\ t\neq 1.$$

Introducing the substitution $t=x^{1/n}$ into the above inequality we infer that

(37) $$\frac{\ln x}{x^{1/n}-1}<\frac{n(1+x^{1/3n})}{x^{1/n}+x^{1/3n}}, \qquad x>0,\ x\neq 1.$$

Multiplying the last inequality by $\frac{x^{1/n}-1}{x-1}$, which is positive for $x>0$, we get

(38) $$\frac{\ln x}{x-1}<\frac{n\,(x^{1/n}-1)\,(1+x^{1/3n})}{(x-1)\,(x^{1/n}+x^{1/3n})}=:C_n(x), \quad n=1, 2, \ldots .$$

Finally we shall prove that the terms of the sequence give $\mathscr{C}$ us stronger bounds of the logarithm-function than the bounds we can get from the sequence $\mathscr{B}$. In this direction let us observe that the corresponding terms of those sequences satisfy the relation

(39) $$C_n(x)=\frac{x^{1/3n}+1}{x^{1/2n}+x^{-1/6n}}\,B_n(x), \qquad n=1, 2, \ldots .$$

So, it remains only to prove that the inequality

(40) $$\frac{x^{1/3n}+1}{x^{1/2n}+x^{-1/6n}}\leqq 1$$

is valid for $x>0$. Indeed, if we take $x^{1/6n}=t$, the last inequality can be writen in the following form

(41) $$(t^3-1)\,(t-1)\geqq 0$$

which obviously holds for every $t>0$. The equality sign occurs if and only if we have $t=1$. Hence, we have

(42) $$C_n(x)<B_n(x), \qquad n=1, 2, \ldots;\ x>0,\ x\neq 1.$$

This proves our theorem. □

If we extend the definition of the function $\frac{\ln x}{x-1}$ at the point $x=1$, by using the analytic continuation, then we can obtain the equality sign in the inequalities from (2).

It is interesting to note that on the basis of the inequalities

(43) $$A_n(x)<\frac{\ln x}{x-1}<B_n(x), \qquad n=1, 2, \ldots;\ x>0,\ x\neq 1,$$

by multiplying with $\frac{x-1}{x^{1/n}-1}$, we can write

(44) $$\frac{2}{x^{1/n}+1}<\frac{\ln x}{n\,(x^{1/n}-1)}<\frac{1}{x^{1/2n}},$$

so that therefrom it follows immediately that

(45) $$\lim_{n\to+\infty} n\,(x^{1/n}-1)=\ln x, \qquad x>0,\ x\neq 1.$$

The first terms $B_1(x)$ and $C_1(x)$ of the considered sequences occur in the papers by J. KARAMATA [2] and D. BLANUŠA [3] where they have proposed the inequalities

(46) $$\frac{\ln x}{x-1}<\frac{1}{\sqrt{x}}=B_1(x), \qquad x>0,\ x\neq 1,$$

and

(47) $$\frac{\ln x}{x-1}<\frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}=C_1(x), \quad x>0,\ x\neq 1.$$

On the other hand, the first term of above sequence $\mathcal{A}$ reduces to the function $\frac{2}{x+1}$ i.e. this first term is reduced to the well known lower bound in the inequality from (6).

Among the properties of the considered sequences we shall turn our attention only to the behaviour of their terms in the case when x converges to $0+$ or 1 or $+\infty$. We think that the behaviour of these function, beside monotony, is of great importance.

By the use of L'HOSPITAL's rule we find

$$\lim_{x\to 0+}\frac{A_n(x)}{x^0}=2n,\quad \lim_{x\to+\infty}\frac{x A_n(x)}{x^0}=2n,\quad \lim_{x\to 1}\frac{A_n(x)-\dfrac{\ln x}{x-1}}{(x-1)^2}=-\frac{1}{12n^2},$$

$$\lim_{x\to 0+}\frac{B_n(x)}{x^{-1/2n}}=n,\quad \lim_{x\to+\infty}\frac{x B_n(x)}{x^{1/2n}}=n,\quad \lim_{x\to 1}\frac{B_n(x)-\dfrac{\ln x}{x-1}}{(x-1)^2}=\frac{1}{24n^2}, \tag{48}$$

$$\lim_{x\to 0+}\frac{C_n(x)}{x^{-1/3n}}=n,\quad \lim_{x\to+\infty}\frac{x C_n(x)}{x^{1/3n}}=n,\quad \lim_{x\to 1}\frac{C_n(x)-\dfrac{\ln x}{x-1}}{(x-1)^4}=\frac{1}{1620n^4}.$$

II

The sequences of approximation functions which should be used for approximation of the logarithmic function, can be constructed sometimes starting from some well-known inequalities for the logarithm-function. But, our opinion is that the above obtained results give us better approximations than the approximations which can be obtained in such a manner. In the following text we will concentrate our attention to such an example.

D. M. SIMEUNOVIĆ proved in [4] the following inequalities

$$\left(\frac{p(b-a)}{b^p-a^p}\right)^{\frac{1}{p-1}}<\frac{\ln b-\ln a}{b-a}<\left(\frac{b^{p-1}-a^{p-1}}{(p-1)(b-a)(ab)^{p-1}}\right)^{\frac{1}{p}},\quad b>a>0;\ p>1. \tag{49}$$

Those inequalities, as well as some another from that paper, can be used to form the minorant or majorant sequences of the function $\frac{\ln x}{x-1}$. If we substitute $b=x(>1)$, $a=1$ in those inequalities and after that we introduce $1/x$ instead of x, we obtain the following inequalities

$$\left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{p-1}}<\frac{\ln x}{x-1}<\left(\frac{x^{p-1}-1}{(p-1)(x-1)x^{p-1}}\right)^{\frac{1}{p}},\quad x>0,\ x\neq 1;\ p>1. \tag{50}$$

The second of the above inequalities is explicitely stated in [4]. For some particular case of p we can obtain the sequences of approximation functions which posses appropriate properties. Now, we shall turn our attention to an example which is selected with respect to its boundary behaviour when $x\to 0+$ or 1 or $+\infty$.

Theorem 2. *Let* $\mathcal{D}:=\{D_k(x)\}$ *and* $\mathcal{E}:=\{E_k(x)\}$, $k=1, 2, \dots$, *be the sequences of functions, defined by*

(51)
$$D_k(x):=\left(\frac{(k+1)(x-1)}{k(x^{(k+1)/k}-1)}\right)^k,$$
$$E_k(x):=\left(\frac{k(x^{1/k}-1)}{(x-1)x^{1/k}}\right)^{k/(k+1)}, \quad k=1, 2, \dots; x>0, x\neq 1.$$

Then, the sequence $\mathcal{D}$ *is increasing and the sequence* $\mathcal{E}$ *is decreasing. Further on the inequalities*

(52)
$$D_k(x)<\frac{\ln x}{x-1}<E_k(x), \qquad k=1, 2, \dots; x>0, x\neq 1,$$

are valid.

Proof. Functions from (51) can be obtained from (50) for $p=(k+1)/k$, $k=1, 2, \dots$, so that the inequalities from (52) are valid on the basis of the results of the paper **[4]**. It remains only to prove the monotonicity of the sequences $\mathcal{D}$ and $\mathcal{E}$.

For a fixed $x\in(0, 1)\cup(1, +\infty)$ let us define the function $\varphi:=\varphi_x(y)$: $(1, 2]\to \mathbf{R}$ by

(53)
$$\varphi(y):=\left(\frac{y(x-1)}{x^y-1}\right)^{1/(y-1)}.$$

Then we have

(54)
$$\varphi\left(\frac{k+1}{k}\right)=D_k(x), \qquad k=1\ 2, \dots .$$

Let $x>1$ be fixed. By a direct calculation we find

(55)
$$\ln\varphi(y)=\frac{1}{y-1}(\ln y+\ln(x-1)-\ln(x^y-1)),$$

wherefrom it follows

(56)
$$\varphi'(y)=(y-1)^{-2}\varphi(y)\psi(y),$$

where

(57)
$$\psi(y)=1-\frac{1}{y}-\frac{x^y(y-1)\ln x}{x^y-1}-\ln y-\ln(x-1)+\ln(x^y-1).$$

The derivative of the function ψ can be written in the form

(58)
$$\psi'(y)=\frac{y-1}{y^2(x^y-1)^2}(y^2x^y\ln^2 x-(x^y-1)^2).$$

Using the substitution $x^y=t$ into the inequality from (6) we find that

(59)
$$y^2x^y\ln^2 x-(x^y-1)^2=t\ln^2 t-(t-1)^2<0$$

(because of the second inequality in (6) we have $\sqrt{t}\ln t<t-1$ for $t>0$). So, in that way we conclude

(60)
$$\psi'(y)<0, \qquad y\in(1, 2].$$

Now, in virtue of (60) and (57), it follows

(61) $$\psi(y)<\psi(1)=0$$

for $y>1$, so that, on the basis of (56) we conclude

(62) $$\varphi'(y)<0.$$

In such a way, since the function φ decreases, and since $(k+1)/k>(k+2)/(k+1)$, we have

(63) $$D_k(x)=\varphi\left(\frac{k+1}{k}\right)<\varphi\left(\frac{k+2}{k+1}\right)=D_{k+1}(x), \qquad x>1.$$

If we introduce $1/x$ instead of x we see that those inequalities hold true also for $x\in(0, 1)$. This proves the monotonicity of the sequence $\mathcal{D}$ i.e. we have

(64) $$D_k(x)<D_{k+1}(x), \qquad k=1, 2, \dots; \ x>0, \ x\neq 1.$$

In order to prove the monotonicity of the sequence we shall consider the function $g:=g_x(y)$: $(0, 1]\to \mathbf{R}$, defined by

(65) $$g(y):=\left(\frac{y(x^y-1)}{(x-1)x^y}\right)^{1/(y+1)}$$

for every fixed $x\in(0, 1)\cup(1, +\infty)$. The function just defined generates the sequence $\mathfrak{E}$ because of

(66) $$g\left(\frac{1}{k}\right)=E_k(x), \qquad k=1, 2, \dots .$$

For every, but fixed, $x>1$ we can write

(67) $$\ln g(y)=\frac{1}{y+1}(\ln y+\ln(x^y-1)-\ln(x-1)-\ln y),$$

wherefrom we have

(68) $$g'(y)=(y+1)^{-2}g(y)h(y),$$

where

(69) $$h(y)=1+\frac{1}{y}+\frac{x^y(y+1)\ln x}{x^y-1}-\ln y-\ln(x^y-1)+\ln\frac{x-1}{x}.$$

By calculation of the derivative of the function h and using the inequality (6), we get

(70) $$h'(y)=-\frac{y+1}{y^2(x^y-1)}(y^2x^y\ln^2 x+(x^y-1)^2)<0.$$

Further, we conclude that h is decreasing and consequently that

(71) $$h(y)>h(1)=2+\frac{(x+1)\ln x}{x-1}>0, \qquad y\in(0, 1].$$

In virtue of (68) that means that

(72) $$g'(y)>0, \qquad y\in(0, 1],$$

i.e. the function g is increasing. Since $1/k > 1/(k+1)$, it follows

(73) $$E_k(x) = g\left(\frac{1}{k}\right) > g\left(\frac{1}{k+1}\right) - E_{k+1}(x), \qquad x > 1.$$

Respecting the above comment in connection with the intervals (0, 1) and $(1, +\infty)$, we conclude in this case also that the following is true

(74) $$E_k(x) > E_{k+1}(x), \qquad k = 1, 2, \ldots;\ x > 0,\ x \neq 1.$$

This proves the monotonicity of the sequence $\mathfrak{E}$. □

At the point $x = 1$ the functions $D_k(x)$, $E_k(x)$ and $\frac{\ln x}{x-1}$ posses the limit value $(=1)$. Hence, the mentioned functions are such that their definition can be completed in such a way at $x = 1$ that the inequalities from (52) are reduced to equalities (at $x = 1$).

For $k = 1$ the inequalities from (52) are reduced on inequalities from (6). Herefrom it can be seen that the sequences $\mathfrak{D}$ and $\mathfrak{E}$ give us bether approximations of logarithmic-function then one which is given by clasical inequalities from (6).

The behaviour of the functions $D_k(x)$ and $E_k(x)$ at $x = 0$ and $x = +\infty$ is given by the following values

(75) $$\lim_{x \to 0+} \frac{D_k(x)}{x^0} = \left(\frac{k+1}{k}\right)^k, \quad \lim_{x \to +\infty} \frac{xD_k(x)}{x^0} = \left(\frac{k+1}{k}\right)^k,$$
$$\lim_{x \to 0+} \frac{E_k(x)}{x^{-1/(k+1)}} = k^{\frac{k+1}{k}}, \quad \lim_{x \to +\infty} \frac{xE_k(x)}{x^{1/(k+1)}} = k^{\frac{k+1}{k}}.$$

Examination of the behaviour of the logarithmic-function with respect to the sequences $\mathfrak{D}$ and $\mathfrak{E}$ in a point $x = 1$, needs some more detailed (i.e. complicated) calculations. Those results we shall formulate in the form of the following

Theorem 3. *The limit values, which are different from zero,*

(76) $$d_k := \lim_{x \to 1} \frac{D_k(x) - \frac{\ln x}{x-1}}{(x-1)^\alpha},$$

and

(77) $$e_k := \lim_{x \to 1} \frac{E_k(x) - \frac{\ln x}{x-1}}{(x-1)^\beta}, \qquad x > 0,\ x \neq 1,$$

exist if and only if $\alpha = \beta = 2$ *and in that case we have*

(78) $$d_k = -\frac{k+1}{24k}, \qquad e_k = \frac{1}{24k}.$$

Proof. In the proof which follows we shall use generalized BERNOULLI's polynomials. For those polynomials the following formula

(79) $$\frac{t^a e^{zt}}{(e^t - 1)^a} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} B_k^{(a)}(z), \qquad |t| < 2\pi,$$

is valid (see for examle [5; pp 18—22]). As it is usual we shall write $B_k^{(a)}(0) \equiv B_k^{(a)}$. Hence, we have $B_k^{(a)} = \frac{d^k}{dt^k}\left\{\left(\frac{t}{e^t-1}\right)^a\right\}\Big|_{t=0}$ so that $B_k^{(a)}$ is a polynomial in a of degree k. For $a=1$ we have (an ordinary) BERNOULLI's polynomials $B_k^{(1)}(z) \equiv B_k(z)$ and therefore for $z=0$ we obtain BERNOULLI's numbers $B_k(0) \equiv B_k$.

Let us denote

$$(80) \qquad f(x) := \frac{\ln x}{x-1}, \quad g_\alpha(x) := \frac{1}{(x-1)^\alpha}, \quad P_{k,\alpha}(x) := \frac{D_k(x) - \dfrac{\ln x}{x-1}}{(x-1)^\alpha}.$$

Using the substitution $\ln x = t$ we can writte

$$(81) \quad f(x) = \frac{t}{e^t-1}, \quad g_\alpha(x) = \frac{1}{t^\alpha}\left(\frac{t}{e^t-1}\right)^\alpha, \quad D_k(x) = \left(\frac{t}{e^t-1}\right)^{-k}\left(\frac{\frac{k+1}{k}t}{e^{\frac{k+1}{k}t}-1}\right)^k.$$

By the use of the formula (76) for $t=0$, we get

$$(82) \qquad \begin{gathered} f(x) = \sum_{\nu=0}^{+\infty} \frac{t^\nu}{\nu!} B_\nu, \quad g_\alpha(x) = \frac{1}{t^\alpha} \sum_{\nu=0}^{+\infty} \frac{t^\nu}{\nu!} B_\nu^{(\alpha)}, \\ D_k(x) = \sum_{\nu=0}^{+\infty} \frac{t^\nu}{\nu!} B_\nu^{(-k)} \cdot \sum_{\nu=0}^{+\infty} \left(\frac{k+1}{k}\right)^\nu \frac{t^\nu}{\nu!} B_\nu^{(k)} = \sum_{\nu=0}^{+\infty} \frac{t^\nu}{\nu!} c_{\nu,k}, \end{gathered}$$

where

$$(83) \qquad c_{\nu,k} = \sum_{j=0}^{\nu} \binom{\nu}{j} \left(\frac{k+1}{k}\right)^j B_{\nu-j}^{(-k)} B_j^{(k)}.$$

On the basis of what we have

$$(84) \qquad P_{k,\alpha}(x) = \big(D_k(x) - f(x)\big) g_\alpha(x) = \frac{1}{t^\alpha} \sum_{\nu=0}^{+\infty} \frac{t^\nu}{\nu!} \gamma_{\nu,k} \cdot \sum_{\nu=0}^{+\infty} \frac{t^\nu}{\nu!} B_\nu^{(\alpha)}$$

where

$$(85) \qquad \gamma_{\nu,k} = c_{\nu,k} - B_\nu.$$

By multiplying the above series we can writte

$$(86) \qquad P_{k,\alpha}(x) = \frac{1}{t^\alpha} \sum_{\nu=0}^{\nu} \frac{t^\nu}{\nu!} \Gamma(\nu, k),$$

where

$$(87) \qquad \Gamma(\nu, k) = \sum_{i=0}^{\nu} \binom{\nu}{i} B_{\nu-i}^{(\alpha)} \gamma_{i,k}.$$

Now, using the tables of the corresponding values of BERNOULLI's polynomials and numbers (see for example [5]), from (82)—(87) we find

$$(88) \qquad \begin{gathered} c_{0,k} = 1, \quad c_{1,k} = -\frac{1}{2}, \quad c_{2,k} = \frac{k-1}{12k}; \\ \gamma_{0,k} = 0, \quad \gamma_{1,k} = 0, \quad \gamma_{2,k} = -\frac{k+1}{12k}; \\ \Gamma(0, k) = 0, \quad \Gamma(1, k) = 0, \quad \Gamma(2, k) = -\frac{k+1}{12k}. \end{gathered}$$

It can be seen now that $\Gamma(2, k)\neq 0$ $(k=1, 2, \ldots)$. In such a way on the basis of (84)—(88) we can write

$$\lim_{x\to 1} P_{k,\alpha}(x)=\lim_{x\to 1}\frac{\sum\limits_{\nu=2}^{+\infty}\frac{t}{\nu!}\Gamma(\nu, k)}{t^{\alpha}}. \tag{89}$$

For the above limit value, to be different from zero, it is neccessary to be $\alpha=2$. In that case we have

$$d_k=\lim_{x\to 1} P_{k,\alpha}(x)=\lim_{x\to 1} P_{k,2}(x)=-\frac{k+1}{24\,k}. \tag{90}$$

In connection with the functions $E_k(x)$ we will proceed the similar method of concluding.

Let us denote

$$g_{\beta}(x)=\frac{1}{(x-1)^{\beta}},\quad Q_{k,\beta}(x)=\frac{E_k(x)-\dfrac{\ln x}{x-1}}{(x-1)^{\beta}}, \tag{91}$$

and let us introduce again the substitution $\ln x=t$. If we make use the development from (78) we can write

$$E_k(x)=e^{-\frac{t}{k+1}}\left(\frac{\frac{t}{k}}{e^{\frac{t}{k}}-1}\right)^{-\frac{k}{k+1}}\left(\frac{t}{e^t-1}\right)^{\frac{k}{k+1}} \tag{92}$$

$$=e^{-\frac{t}{k+1}}\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!\,k^{\nu}}B_{\nu}\left(-\frac{k}{k+1}\right)\cdot\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}B_{\nu}\left(\frac{k}{k+1}\right)=e^{-\frac{t}{k+1}}\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}m_{\nu,k},$$

where

$$m_{\nu,k}=\sum_{j=0}^{+\infty}\binom{\nu}{j}\frac{1}{k^j}B_j\left(-\frac{k}{k+1}\right)B_{\nu-j}\left(\frac{k}{k+1}\right). \tag{93}$$

In the end, by the use of the development of the function $e^{-\frac{k}{k+1}}$ definitely we have

$$E_k(x)=\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}\frac{(-1)^{\nu}}{(k+1)^{\nu}}\cdot\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}m_{\nu,k}=\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}\mu_{\nu,k}, \tag{94}$$

where

$$\mu_{\nu,k}=\sum_{i=0}^{\nu}\binom{\nu}{i}m_{i,k}\frac{(-1)^{\nu-i}}{(k+1)^{\nu-i}}. \tag{95}$$

Now we obtain

$$Q_{k,\beta}(x)=\big(E_k(x)-f(x)\big)g_{\beta}(x)=\frac{1}{t^{\beta}}\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}(\mu_{\nu,k}-B_{\nu})\cdot\sum_{\nu=0}^{+\infty}\frac{t^{\nu}}{\nu!}B_{\nu}^{(\beta)}=\frac{1}{t^{\beta}}\sum_{\nu=0}^{+\infty}\frac{t_{\nu}}{\nu!}M(\nu, k),$$

where we have

$$M(\nu, k)=\sum_{j=0}^{\nu}\binom{\nu}{j}(\mu_{j,k}-B_j)B_{\nu-j}^{(\beta)}. \tag{97}$$

Using again the tables of BERNOULLI's polynomials from (93)—(97), we calculate

$$(98)\quad \begin{aligned} &m_{0,k}=1, \quad m_{1,k}=\frac{-k+1}{2(k+1)}, \quad m_{2,k}=\frac{2k^3-7k^2+4k+1}{12k(k+1)^2};\\ &\mu_{0,k}=0, \quad \mu_{1,k}=-\frac{1}{2}, \quad \mu_{2,k}=\frac{2k+1}{12k};\\ &M(0,k)=0, \quad M(1,k)=0, \quad M(2,k)=\frac{1}{12k}. \end{aligned}$$

Since $M_{2,k}\neq 0$ ($k=1, 2, \ldots$), we can write further on that

$$(99)\quad Q_{k,\beta}(x)=\frac{1}{t^{\beta}}\left(\frac{t^2}{2!}\frac{1}{12k}+\frac{t^3}{3!}M(3,k)+\ldots\right).$$

For $\beta=2$ we will have $\lim\limits_{x\to 1} Q_{k,\beta}(x)\neq 0$ and in that case we find

$$(100)\quad e_k=\lim_{x\to 1} Q_{k,\beta}(x)=\lim_{x\to 1} Q_{k,2}(x)=\frac{1}{24k}. \quad \square$$

On the basis of (48), (75) and (78) it follows tha the sequences $\mathscr{A}$ and $\mathscr{C}$ gives us better approximation of the logarithmic function at the points 0, 1, $+\infty$ then the sequences $\mathscr{B}$, $\mathscr{D}$, $\mathscr{E}$.

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O APROKSIMACIJI LOGARITAMSKE FUNKCIJE NIZOVIMA ALGEBARSKIH FUNKCIJA (I)

I. Lazarević i *A. Lupaş*

Karakteristični tok logaritamske funkcije onemogućava njenu dobru jednostrnu aproksimaciju na (0, $+\infty$) pomoću funkcija iz uobičajenih klasa. U ovom radu posmatra se aproksimacija logaritamske funkcije izvesnim nizovima algebarskih funkcija (nizovi $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$). Dokazuje se monotonija ovih nizova. Na osnovu toga jednostranom aproksimacijom sa donje odnosno gornje strane vrši se ukliještenje logaritamske krive na celom njenom domenu definisanosti. Nadalje dobijeni su rezultati asimptotskog ponašanja pomenute aproksimacije u karakterističnim tačkama logaritamske funkcije.