UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 735 — № 762 (1982), 35—38.

741. NOTE ON A VECTOR NORM INEQUALITY*

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1. BETH and VAN DER CORPUT have shown that if $\lambda \geqq 2$ and z and w are complex numbers, then (see [**1**, p. 322]):

$$|z+w|^\lambda + |z-w|^\lambda \geqq 2(|z|^\lambda + |w|^\lambda).$$

This inequality was extended by KLAMKIN [**2**] who showed that

$$S_\lambda \equiv \left(\sum | \pm V_1 \pm \cdots \pm V_n|^\lambda\right)^{1/\lambda} \geqq 2^{n/\lambda} \left(\sum_{i=1}^{n} |V_i|^2\right)^{1/2} \tag{1}$$

twhere $\lambda > 2$, V_i are vectors in E^s and the summation on the left-hand side is aken over all 2^n possible choices of the $\pm$ signs. The inequality is reversed for $\lambda < 2$ $(\neq 0)$, while for $\lambda = 0,\ 2$ there is identity.

It is also shown that for $\lambda \geqq 2$

$$S_\lambda^\lambda \geqq 2^n \sum_{i=1}^{n} |V_i|^\lambda. \tag{2}$$

In this note we shall prove the following extension of (1) and (2)

Theorem 1. (i) *If* $p,\ \lambda \geqq 2$, *then*

$$2^{\frac{n}{\lambda}} Q_p \leqq S_\lambda \leqq n^{\frac{1}{2}-\frac{1}{p}}\, 2^{\frac{n-1}{2}+\frac{1}{\lambda}} Q_p, \tag{3}$$

where $Q_p = \left(\sum_{i=1}^{n} |V_i|^p\right)^{1/p}$. *If* $0 \leqq p,\ \lambda \leqq 2$ *then the reverse inequalities in* (3) *are valid.*

(ii) *If* $0 < \lambda \leqq 2,\ p \geqq 2$, *then*

$$2^{\frac{n-1}{2}+\frac{1}{\lambda}} Q_p \leqq S_\lambda \leqq 2^{\frac{n}{\lambda}}\, n^{\frac{1}{2}-\frac{1}{p}} Q_p. \tag{4}$$

For $0 < p \leqq 2$ *and* $\lambda \geqq 2$ *the reverse inequalities in* (4) *are valid.*

Proof. (i) Let $\lambda,\ p > 2$. Since $2^{-n} \sum | \pm V_1 \pm \cdots \pm V_n|^2 = \sum_{i=1}^{n} |V_i|^2$ using the JENSEN inequality for convex function $x \mapsto x^\lambda$, we have

$$2^{-n} \sum |\pm V_1 \pm \cdots \pm V_n|^\lambda \geqq \left(2^{-n} \sum |\pm V_1 \pm \cdots \pm V_n|^2\right)^{\frac{\lambda}{2}} = \left(\sum_{i=1}^{n} |V_i|^2\right)^{\frac{\lambda}{2}} \tag{5}$$

* Presented by M. S. KLAMKIN.
Ovaj rad je finansirala Republička Zajednica Nauka Srbije.

i.e. $S_\lambda \geqq 2^{n/\lambda} Q_2$. Now, using the inequality of sums of order t: $Q_2 \geqq Q_p$ $(p \geqq 2)$, we have the first inequality of (3).

Now, we shall prove the second inequality. Since

$$\sum |\pm V_1 \pm \cdots \pm V_n|^\lambda = 2 \sum |V_1 \pm V_2 \pm \cdots \pm V_n|^\lambda,$$

where the summation on the right-hand side is taken over all 2^{n-1} permutations of the $\pm$ signs, we have

$$S_\lambda = 2^{\frac{1}{\lambda}} (\sum |V_1 \pm \cdots \pm V_n|^\lambda)^{\frac{1}{\lambda}} \leqq 2^{\frac{1}{\lambda}} (\sum |V_1 \pm \cdots \pm V_n|^2)^{\frac{1}{2}} = 2^{\frac{1}{\lambda}-\frac{1}{2}} S_2 = 2^{\frac{n-1}{2}+\frac{1}{\lambda}} Q_2$$

$$= 2^{\frac{n-1}{2}+\frac{1}{\lambda}} n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=1}^{n} |V_i|^2\right)^{\frac{1}{2}} \leqq 2^{\frac{n-1}{2}+\frac{1}{\lambda}} n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=1}^{n} |V_i|^p\right)^{\frac{1}{p}} = 2^{\frac{n-1}{2}+\frac{1}{\lambda}} n^{\frac{1}{2}-\frac{1}{p}} Q_p.$$

Analogously, we can prove that the reverse inequalities in (4) hold if $0 \leqq p, \lambda \leqq 2$.

(ii) Now, let $0 < \lambda \leqq 2$, $p \geqq 2$. Then we have the reverse inequalities in (5). So,

$$S_\lambda \leqq 2^{\frac{n}{\lambda}} Q_2 = 2^{\frac{n}{\lambda}} n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=1}^{n} |V_i|^2\right)^{\frac{1}{2}} \leqq 2^{\frac{n}{\lambda}} n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=1}^{n} |V_i|^p\right)^{\frac{1}{p}} = 2^{\frac{n}{\lambda}} n^{\frac{1}{2}-\frac{1}{p}} Q_p$$

and

$$S_\lambda = 2^{\frac{1}{\lambda}} (\sum |V_1 \pm \cdots \pm V_n|^\lambda)^{\frac{1}{\lambda}} \geqq 2^{\frac{1}{\lambda}} (\sum |V_1 \pm \cdots \pm V_n|^2)^{\frac{1}{2}}$$

$$= 2^{\frac{1}{\lambda}} \left(2^{n-1} \sum_{i=1}^{n} |V_i|^2\right)^{\frac{1}{2}} = 2^{\frac{n-1}{2}+\frac{1}{\lambda}} Q_2 \geqq 2^{\frac{n-1}{2}+\frac{1}{\lambda}} Q_p.$$

Analogously, we can prove that the reverse inequalities are valid in (4) if $0 < p \leqq 2$ and $\lambda \geqq 2$.

2. T. R. Shore [3] proved the following generalizations of (1) and (2):

For $\lambda \geqq 2$,

$$\sum \|\pm f_1 \pm \cdots \pm f_n\|_2^\lambda \geqq 2^n \left(\sum_{i=1}^{n} \|f_i\|_2^2\right)^{\lambda/2} \geqq 2^n \sum_{i=1}^{n} \|f_i\|_2^\lambda, \tag{6}$$

where $f_1, \ldots, f_n$ are real-valued functions in $L^2(X, \mathcal{B}, \mu)$, where μ is σ-finite.

We shall give extensions of these results in the case when $f_1, \ldots, f_n$ are real-valued functions in $L^p(X, \mathcal{B}, \mu)$ is σ-finite. We shall use the following notations

$$R_\lambda = (\sum \|\pm f_1 \pm \cdots \pm f_n\|_p^\lambda)^{\frac{1}{\lambda}} \quad \text{and} \quad T_q = \left(\sum_{i=1}^{n} \|f_i\|_p^q\right)^{\frac{1}{q}}.$$

Theorem 2. (i) *Let* $2 \leqq p \leqq \lambda$. *If* $p \leqq q$, *then*

$$2^{\frac{n}{\lambda}} T_q \leqq R_\lambda \leqq n^{\frac{1}{2}-\frac{1}{q}} 2^{\frac{n-1}{2}+\frac{1}{\lambda}} T_q, \tag{7}$$

and if $p \geqq q$, *then*

$$2^{\frac{n}{\lambda}} n^{\frac{1}{p}-\frac{1}{q}} T_q \leqq R_\lambda \leqq n^{\frac{1}{2}-\frac{1}{p}} 2^{\frac{n-1}{2}+\frac{1}{\lambda}} T_q. \tag{8}$$

Let $0<\lambda\leqq p$ *and* $1\leqq p\leqq 2$. *If* $p\geqq q$ *then the reverse inequalities in* (7) *hold, and if* $p\leqq q$ *then the reverse inequalities in* (8) *hold.*

(ii) *Let* $\lambda\geqq p$, $1\leqq p\leqq 2$. *If* $q\geqq p$, *then*

$$n^{\frac{1}{2}-\frac{1}{p}}2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}}T_q\leqq R_\lambda\leqq n^{\frac{1}{p}-\frac{1}{q}}2^{\frac{n-1}{p}+\frac{1}{\lambda}}T_q, \tag{9}$$

and if $q\leqq p$ *then*

$$n^{\frac{1}{2}-\frac{1}{q}}2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}}T_q\leqq R_\lambda\leqq 2^{\frac{n-1}{p}+\frac{1}{\lambda}}T_q. \tag{10}$$

Let $0<\lambda\leqq p$ *and* $p\geqq 2$. *If* $p\geqq q$ *then the reverse inequalities in* (9) *hold, and if* $p\leqq q$ *then the reverse inequalities in* (10) *are valid.*

Proof. (i) Let $\lambda\geqq p\geqq 2$. Then

$$2^{-n}\sum\|\pm f_1\pm\cdots\pm f_n\|_p^\lambda\geqq\left(2^{-n}\sum\|\pm f_1\pm\cdots\pm f_n\|_p^p\right)^{\frac{\lambda}{p}}$$
$$=\Big(2^{-n}\sum\int_X|\pm f_1\pm\cdots\pm f_n|^p\,\mathrm{d}\mu\Big)^{\frac{\lambda}{p}}=\Big(\int_X 2^{-n}\sum|\pm f_1\pm\cdots\pm f_n|^p\,\mathrm{d}\mu\Big)^{\frac{\lambda}{p}}$$
$$\geqq\Big(\int_X\sum_{i=1}^n|f_i|^p\,\mathrm{d}\mu\Big)^{\frac{\lambda}{p}}=\Big(\sum_{i=1}^n\|f_i\|_p^p\Big)^{\frac{\lambda}{p}},$$

i.e., for $q\geqq p$, $R_\lambda\geqq 2^{n/\lambda}T_p\geqq 2^{n/\lambda}T_q$, and for $q\leqq p$,

$$R_\lambda\geqq 2^{\frac{n}{\lambda}}T_p=2^{\frac{n}{\lambda}}n^{\frac{1}{p}}\Big(\frac{1}{n}\sum_{i=1}^n\|f_i\|_p^p\Big)^{\frac{1}{p}}\geqq 2^{\frac{n}{\lambda}}n^{\frac{1}{p}}\Big(\frac{1}{n}\sum_{i=1}^n\|f_i\|_p^q\Big)^{\frac{1}{q}}=2^{\frac{n}{\lambda}}n^{\frac{1}{p}-\frac{1}{q}}T_q.$$

We also have

$$R_\lambda=2^{\frac{1}{\lambda}}\left(\sum\|f_1\pm\cdots\pm f_n\|_p^\lambda\right)^{\frac{1}{\lambda}}\leqq 2^{\frac{1}{\lambda}}\left(\sum\|f_1\pm\cdots\pm f_n\|_p^p\right)^{\frac{1}{p}}$$
$$=2^{\frac{1}{\lambda}-\frac{1}{p}}R_p=2^{\frac{1}{\lambda}-\frac{1}{p}}\Big(\int_X\sum|\pm f_1\pm\cdots\pm f_n|^p\,\mathrm{d}\mu\Big)^{\frac{1}{p}}.$$

Using the second inequality in (3) for $\lambda=p$, we have

$$R_\lambda\leqq n^{\frac{1}{2}-\frac{1}{p}}2^{\frac{n-1}{2}+\frac{1}{p}}\Big(\int_X\sum_{i=1}^n|f_i|^p\,\mathrm{d}\mu\Big)^{\frac{1}{p}}=n^{\frac{1}{2}-\frac{1}{p}}2^{\frac{n-1}{2}+\frac{1}{p}}T_p.$$

Now, for $p\geqq q$, we obtain $R_\lambda\leqq n^{\frac{1}{2}-\frac{1}{p}}2^{\frac{n-1}{2}+\frac{1}{p}}T_q$, and for $p\leqq q$,

$$R_\lambda\leqq n^{\frac{1}{2}}2^{\frac{n-1}{2}+\frac{1}{p}}\Big(\frac{1}{n}\sum_{i=1}^n\|f_i\|_p^p\Big)^{\frac{1}{p}}\leqq n^{\frac{1}{2}-\frac{1}{q}}2^{\frac{n-1}{2}+\frac{1}{p}}T_q.$$

Analogously, we can prove the reverse results.

(ii) Let $\lambda\geqq p$, $1\leqq p\leqq 2$. Then

$$2^{-n}\sum\|\pm f_1\pm\cdots\pm f_n\|_p^\lambda\geqq\left(2^{-n}\sum\|\pm f_1\pm\cdots\pm f_n\|_p^p\right)^{\frac{\lambda}{p}},$$

i.e.

$$R_\lambda \geqq 2^{\frac{n}{\lambda}-\frac{n}{p}} \Big(\int_X \sum | \pm f_1 \pm \cdots \pm f_n |^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}}.$$

Using (3), we have

$$R_\lambda \geqq n^{\frac{1}{2}-\frac{1}{p}} 2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}} \Big(\int_X \sum_{i=1}^{n} |f_i|^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} = n^{\frac{1}{2}-\frac{1}{p}} 2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}} T_p.$$

Now, if $q \geqq p$, we obtain $R_\lambda \geqq n^{\frac{1}{2}-\frac{1}{p}} 2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}} T_q$, and if $q \leqq p$,

$$R_\lambda \geqq n^{\frac{1}{2}} 2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}} \Big(\frac{1}{n} \sum_{i=1}^{n} \|f_i\|_p^p \Big)^{\frac{1}{p}} \geqq n^{\frac{1}{2}-\frac{1}{q}} 2^{(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+\frac{n}{\lambda}} T_q.$$

We also have

$$R_\lambda \leqq 2^{\frac{1}{\lambda}-\frac{1}{p}} R_p = 2^{\frac{1}{\lambda}-\frac{1}{p}} \Big(\int_X \sum | \pm f_1 \pm \cdots \pm f_n |^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}}$$

$$= 2^{\frac{n-1}{p}+\frac{1}{\lambda}} \Big(\int_X \sum_{i=1}^{n} |f_i|^p \, \mathrm{d}\mu \Big)^{\frac{1}{p}} = 2^{\frac{n-1}{p}+\frac{1}{\lambda}} T_p,$$

so, for $q \geqq p$, we obtain

$$R_\lambda \leqq 2^{\frac{n-1}{p}+\frac{1}{\lambda}} T_p = 2^{\frac{n-1}{p}+\frac{1}{\lambda}} n^{\frac{1}{p}} \Big(\frac{1}{n} \sum_{i=1}^{n} \|f_i\|_p^p \Big)^{\frac{1}{p}} = 2^{\frac{n-1}{p}+\frac{1}{\lambda}} n^{\frac{1}{p}-\frac{1}{q}} T_q,$$

and for $q \leqq p$, $R_\lambda \leqq 2^{\frac{n-1}{p}+\frac{1}{\lambda}} T_p \leqq 2^{\frac{n-1}{p}+\frac{1}{\lambda}} T_q$.

Similarly, we can prove the reverse results.

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NOTA O JEDNOJ NEJEDNAKOSTI ZA NORME VEKTORA

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U radu su data neka proširenja rezultata M. S. Klamkina iz [2] i T. R. Shorea iz [3].