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739. A GENERALIZATION OF CHEBYSHEV INEQUALITY FOR CONVEX SEQUENCES OF ORDER k*

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In this paper, using the monotony of convex sequences of order $k\,(\geqq 2)$, we will prove a generalization of CHEBYSHEV inequality (see [1] and [2]).

Let S_k be a set of all real sequences $a=(a_1, \dots, a_n)$ which are convex of order k $(1<k<n)$, i. e.

$$S_k=\left\{a \mid \Delta^k a_m=\sum_{i=0}^{k}(-1)^i\binom{k}{i}a_{m+k-i}\geqq 0 \quad (1\leqq m\leqq n-k)\right\}.$$

Define the sequence $a^{(r)}=(a_1^{(r)}, \dots, a_n^{(r)})$ (r is a fixed positive integer) recursively by

$$a_m^{(r)}=\frac{1}{m}a_m^{(r-1)},\quad a_m^{(1)}=a_m \quad \left(a_m^{(r)}=\frac{a_m}{m^{r-1}}\right).$$

Let $S_k^{(p)}=\{a \mid a\in S_k \wedge \Delta^{k-i}a_1^{(i+1)}\geqq 0 \ (i=1, \dots, p))\}$, where $p<k$.

The following results are proved in [3].

Theorem A. *The implication* $a\in S_k^{(1)} \Rightarrow a^{(2)}\in S_{k-1}$ $(k\geqq 2)$ *holds.*

Theorem B. *The implication* $a\in S_k^{(k-1)} \Rightarrow a^{(k)}\in S_1$ $(k\geqq 2)$ *holds.*

The following generalization of CHEBYSHEV inequality is proved in [4].

Theorem C. *If* $a=(a_1, \dots, a_n)$ *and* $b=(b_1, \dots, b_n)$ *are two real sequences such that*

$$0=a_1\leqq \cdots \leqq a_n, \qquad 0=b_1\leqq \cdots \leqq b_n,$$

$$\left.\begin{array}{l} a_{i-1}-2a_i+a_{i+1}\geqq 0 \\ b_{i-1}-2b_i+b_{i+1}\geqq 0 \end{array}\right. \qquad (i=2, \dots, n-1),$$

then the following inequality is valid

$$C_n(a, b; p)\geqq C_{n-1}(a, b; p), \tag{1}$$

where

$$C_n(a, b; p)=\left(\sum_{i=1}^{n}p_i(i-1)\right)^2\sum_{i=1}^{n}p_i a_i b_i-\sum_{i=1}^{n}p_i(i-1)^2\sum_{i=1}^{n}p_i a_i\sum_{i=1}^{n}p_i b_i$$

and $p=(p_1, \dots, p_n)$ *is a positive sequence.*

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From the inequality (1), as shown in [4], we obtain directly the following inequality

$$\left(\sum_{i=1}^{n} p_i(i-1)\right)^2 \sum_{i=1}^{n} p_i a_i b_i \geqq \sum_{i=1}^{n} p_i (i-1)^2 \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \tag{2}$$

representing a generalization of CHEBYSHEV inequality for convex sequences.

Using the properties of convex sequences of order $k(\geqq 2)$ proved in Theorem A and B, we will prove the following result:

Theorem 1. *Let $p=(p_1, \dots, p_n)$ be a positive sequence. If r positive sequences $a=(a_1, \dots, a_n)$, $b=(b_1, \dots, b_n)$, $\dots$, $e=(e_1, \dots, e_n)$ belong to a set $S_k^{(k-1)}$ $(n\geqq k)$, then the inequality*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i b_i \cdots e_i \geqq N_{r,k} \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \cdots \sum_{i=1}^{n} p_i e_i \tag{3}$$

holds, where

$$N_{r,k} = \frac{\left(\sum_{i=1}^{n} p_i\right)^{r-1} \left(\sum_{i=1}^{n} p_i i^{r(k-1)}\right)}{\left(\sum_{i=1}^{n} p_i i^{k-1}\right)^r} \geqq 1.$$

Proof. If in CHEBYSHEV inequality for r sequences

$$\left(\sum_{i=1}^{n} q_i\right)^{r-1} \sum_{i=1}^{n} q_i x_i y_i \cdots t_i \geqq \sum_{i=1}^{n} q_i x_i \sum_{i=1}^{n} q_i y_i \cdots \sum_{i=1}^{n} q_i t_i \tag{4}$$

we substitute $r=2$, $q_i=p_i i^{k-1}$, $x_i=i^{(k-1)(r-1)}$, $y_i=\frac{a_i b_i \cdots e_i}{i^{r(k-1)}}$ $(i=1, \dots, n)$, we will obtain

$$\sum_{i=1}^{n} p_i i^{k-1} \sum_{i=1}^{n} p_i a_i b_i \cdots e_i \geqq \sum_{i=1}^{n} p_i i^{r(k-1)} \sum_{i=1}^{n} p_i \frac{a_i b_i \cdots e_i}{i^{(r-1)(k-1)}}. \tag{5}$$

With new substitutions $q_i=p_i i^{k-1}$, $x_i=\frac{a_i}{i^{k-1}}$, $y_i=\frac{b_i}{i^{k-1}}$, $\dots$, $t_i=\frac{e_i}{i^{k-1}}$ $(i=1,\dots, n)$, the inequality (4) becomes

$$\left(\sum_{i=1}^{n} p_i i^{k-1}\right)^{r-1} \sum_{i=1}^{n} p_i \frac{a_i b_i \cdots e_i}{i^{(r-1)(k-1)}} \geqq \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \cdots \sum_{i=1}^{n} p_i e_i. \tag{6}$$

From the inequalities (5) and (6), the inequality (3) follows.

Since the sequences $a=(a_1, \dots, a_n)$, $b=(b_1, \dots, b_n)$, $\dots$, $e=(e_1, \dots, e_n)$ belong to the set $S_k^{(k-1)}$, then the sequences

$$a^{(k)}=\left(\frac{a_i}{i^{k-1}}\right), \quad b^{(k)}=\left(\frac{b_i}{i^{k-1}}\right), \dots, \quad e^{(k)}=\left(\frac{e_i}{i^{k-1}}\right),$$

according to the Theorem A and B, are monotonically increasing sequences. This means that the introduced substitutions satisfy the conditions necessary for inequality (4).

Let us show that $N_{r,k}$, for fixed r, increases as the parameter k increases, i. e. inequality (3) becomes sharper as k increases.

Inequality

$$\left(\frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i b_i^r}\right)^{1/r} \geqq \left(\frac{\sum_{i=1}^{n} p_i a_i^s}{\sum_{i=1}^{n} p_i b_i^s}\right)^{1/s} \qquad \left(\begin{matrix} r \geqq s, \ r, \ s \neq 0, \\ |r| < +\infty, \ |s| < +\infty \end{matrix}\right)$$

which holds if:

$$p_1 > 0, \ldots, p_n > 0; \ a_1 > 0, \ldots, a_n > 0, \ b_1 \geqq \cdots \geqq b_n > 0, \ \frac{b_1}{a_1} \leqq \cdots \leqq \frac{b_n}{a_n}$$

or

$$p_1 > 0, \ldots, p_n > 0; \ a_1 > 0, \ldots, a_n > 0, \ b_n \geqq \cdots \geqq b_1 > 0, \ \frac{b_1}{a_1} \geqq \cdots \geqq \frac{b_n}{a_n}$$

(see, e. g. [**1**], [**5**], [**6**]) for $s=1$, $a_i = i^{k-1}$, $b_i = i^{k-2}$ $(i=1, \ldots, n)$, becomes $N_{r,k} \geqq N_{r,k-1}$, wherefrom we conclude that

$$N_{r,k} \geqq N_{r,k-1} \geqq \cdots \geqq N_{r,1} = 1.$$

Let us quote same corrolaries of the Theorem 1.

Corrolary 1. *Let* $a=(a_1, \ldots, a_n)$, $b=(b_1, \ldots, b_n)$ *be positive convex sequences with the property* $a_2 - a_1 \geqq 0$, $b_2 - b_1 \geqq 0$. *Then, if* $p=(p_1, \ldots, p_n)$ *is positive sequence, then the inequality*

$$\left(\sum_{i=1}^{n} p_i i\right)^2 \sum_{i=1}^{n} p_i a_i b_i \geqq \sum_{i=1}^{n} p_i i^2 \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i, \tag{7}$$

holds.

This inequality is connected with inequality (2), and can be derived from the inequality (3) for $k=r=2$.

Corrolary 2. *For* $k=2$ *and* $p_i = 1$ $(i=1, \ldots, n)$ *the inequality* (3) *becomes*

$$\sum_{i=1}^{n} a_i b_i \cdots e_i \geqq M(r) \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \cdots \sum_{i=1}^{n} e_i \tag{8}$$

where

$$M(r) = \frac{2^r \sum_{i=1}^{n} i^r}{n^r (n+1)^r}.$$

For instance

$$M(2) = \frac{2(2n+1)}{3n(n+1)}, \quad M(3) = \frac{2}{n(n+1)}, \quad M(4) = \frac{8(2n+1)(3n^2+3n-1)}{15n^3(n+1)^3}.$$

Since $\lim_{n \to +\infty} (n^{r-1} M(r)) = \frac{2^r}{r+1}$, applying boundary value $(n \to +\infty)$ inequality (8) can be reduced to Andersson inequality (see [**1**], [**7**]),

$$\int_0^1 f_1(x) \cdots f_r(x) \, dx \geqq \frac{2^r}{r+1} \int_0^1 f_1(x) \, dx \cdots \int_0^1 f_r(x) \, dx.$$

REFERENCES

1. D. S. MITRINOVIĆ (In cooperation with P. M. VASIĆ): *Analytic inequalities*. Berlin—Heidelberg—New York, 1970.
2. D. S. MITRINOVIĆ and P. M. VASIĆ: *History variations and generalizations of the Čebyšev inequality and question of some priorities*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **461**—№ **497** (1974), 1—30.
3. I. Ž. MILOVANOVIĆ and G. V. MILOVANOVIĆ: *Some properties of convex sequences of order k*. (In prep.)
4. P. M. VASIĆ and R. Ž. ĐORĐEVIĆ: *Čebyšev inequality for convex sets*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **412**—№ **460** (1973), 17—20.
5. A. W. MARSALL, I. OLKIN, F. PROSCHAN: *Monotonocity of ratios of means and other applications of majorization*. In Inequalities edited by O. Shisha. New York—London 1967, 177—190.
6. P. M. VASIĆ and I. Ž. MILOVANOVIĆ: *On the ratio of means*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № **577**—№ **598** (1977), 33—37.
7. B. J. ANDERSSON: *An inequality for convex functions*. Nordisk. Math. Tidsk. **6** (1958), 25—26.

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JEDNA GENERALIZACIJA ČEBIŠEVLJEVE NEJEDNAKOSTI ZA KONVEKSNE NIZOVE REDA k

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U ovom radu, koristeći se osobinama monotonosti nizova konveksnih reda k ($\geqq 2$,), dobijamo jednu generalizaciju ČEBIŠEVljeve nejednakosti. Ukazano je i na jedan integralni analogon.