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738. ON THE BEHAVIOUR OF SEQUENCES OF LEFT AND RIGHT DERIVATIVES OF A CONVERGENT SEQUENCE OF CONVEX FUNCTIONS*

Ivan B. Lacković

Supposing that a sequence of convex functions converges pointwise in an interval I to a finite function, we have, in this paper, considered the convergence of left and right derivatives. It is shown that convexity is sufficient to ensure a relatively good behaviour of the sequences of derivatives in the interior points, as well as in the boundary points of the considered domain. It is also shown that this convergence point by point can be, in theorems of this sort, replaced by weaker assumptions.

1. It is well known that if a sequence $f_n (n \in \mathrm{N})$ of real functions converges point by point to a real function f, then the sequence of derivatives $f_n' (n \in \mathrm{N})$ need not converge to f'. Simple examples of such sequences are readily constructed. On the other hand, it is well known that rather strong conditions have to be imposed to ensure that $f_n \to f (n \to +\infty)$ implies $f_n' \to f'$.

Besides the standard condition of uniform convergence, other conditions which ensure the validity of the mentioned implication are also known. A class of sequences which allows the above implication is the class of sequences $f_n (n \in \mathrm{N})$, where each function f_n is convex on a certain domain.

In this paper we shall investigate the behaviour of the sequences of derivatives of convergent sequences of convex functions. Some results of that sort, as it will be shown later, are known in literature, so that in this paper we will start from those known facts and we will give some generalizations, additions and comments in connection with the problem in consideration.

In the present paper, by I we will denote one of the intervals

$$[a, b] (-\infty < a < b < +\infty), \quad (a, b) \quad (-\infty \leqslant a \leqq b \leqq +\infty), \quad [a, +\infty) \text{ or } (-\infty, a],$$

where $-\infty < a < +\infty$.

For a function $f: I \to R$ it is said that it is convex on I if $f(tx + (1-t)y) \leqq tf(x) + (1-t)f(y)$ holds true for all $x, y \in I$ and all $t \in (0, 1)$. If for all $x, y \in I (x \neq y)$ and all $t \in (0, 1)$ the inequality $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ holds then for the function f we shall say that it is strictly convex on I. As it is well known a function f which is convex or strictly convex possesses many of the „good" properties on I.

Therefrom it follows that the sequences of functions, convex (or strictly convex) on the same domain I, have also „good" properties which sequences

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of arbitrary functions need not possess. If in addition to the convexity of all considered functions of a sequence, we consider sequences which have some other properties (for example the property of boundedness, convergence, monotony etc.) then we obtain classes of sequences of much wealthier structures.

A theorems of such nature was proved by MASATSUGU TSUJI in [1]. His theorem reads:

Theorem 1.1. *Let f and $f_n (n \in \mathrm{N})$ be convex functions of the variable $x \in [a, b]$, such that we have $\lim\limits_{n \to +\infty} f_n(x) = f(x)$ for all $x \in [a, b]$. Suppose that at some point $x_0 \in (a, b)$ there exist the derivatives f' and f_n' for all $n \in \mathrm{N}$. Then we have $\lim\limits_{n \to +\infty} f_n'(x_0) = f'(x_0)$.*

The well known fact is, that if f is a convex function on I, then there exists the derivative f' on I excepting possibly on a countable set.

For a given sequence, where all the functions are defined on the same domain I, let us denote by $G_n (n \in \mathrm{N})$ those subsets of I in which the functions f_n respectively do not have the first order derivative. If those functions $f_n (n \in \mathrm{N})$ are convex on I and if the functions f defined by $f(x) = \lim\limits_{n \to +\infty} f_n(x)$ is finite on I, then the function f is also convex on I. Let us denote by G_0 the subset of I in which the function f does not have the first order derivative. Under the above suppositions the set $G = \bigcup\limits_{k=0}^{+\infty} G_k$ is at most countable set.

For the above reasons and by using the above notations, theorem 1.1. can be formulated in the following way:

Theorem 1.2. *If the sequence $f_n (n \in \mathrm{N})$ of convex functions converges point by point everywhere in I to the finite function f, then we have $\lim\limits_{n \to +\infty} f_n'(x) = f'(x)$ for all $x \in I \setminus G$.*

Such formulation of the theorem 1.1. can be found in the monograph [2] p. 21. Let us point out that in a particular case the above set G can be void.

2. In this part of the present paper we will give a generalization of the above theorems as well as a few observations which, as it seems to us, describe completely the behaviour of the sequence of derivatives of a convergent sequence of convex functions. This generalization is based on the fact that if f is a convex function on I then there exist the left and right derivatives of the function f everywhere in the interval I, where the right derivative of f exists also at the left end point of I, if this point belongs to I, and analogously left derivative of f exists at the right end point of I if this point belongs to I. Those derivatives can be respectively $+\infty$ or $-\infty$ at those end points. In such a way we can say that the left and right derivatives exist in the extended sense.

If $f_n (n \in \mathrm{N})$ is a given sequence of convex functions, then bu f'_{n+} and by f'_{n-} we will denote the right and the left derivative of f_n respectively. In a similar way by f'_+ and f'_- we will denote the right and left derivatives of the function f. By using those notations we will prove the following statement.

Lemma 2.1. *There exists a sequence $f_n (n \in \mathrm{N})$ of functions, defined on* $[-1, 1]$, *which satisfies the following conditions:*

(a) *all the functions of the sequence $f_n (n \in \mathrm{N})$ are convex on* $[-1, 1]$,

(b) *the sequence $f_n (n \in \mathrm{N})$ converges uniformly in* $[-1, 1]$ *to the finite (convex) function f,*

and this sequence satisfies one of the following conditions:

(c) *the sequence $f'_{n+}(0)$ of right derivatives (or the sequence $f'_{n-}(0)$ of left derivatives) is convergent, but the limit value of that sequence is different from $f'_+(0)$ (or different from $f'_-(0)$),*

(d) *the sequence $f'_{n+}(0) (n \in \mathrm{N})$ (i. e. the sequence $f'_{n-}(0)$) is not convergent.*

Proof. To prove the above statement it is sufficient to consider two sequences $u_n (n \in \mathrm{N})$ and $v_n (n \in \mathrm{N})$ of functions defined on $[-1, 1]$ in the following way

$$u_n(x) = \begin{cases} \frac{n}{2} x^2 + \frac{1}{2n}, & |x| \leqq \frac{1}{n} \\ |x|, & |x| > \frac{1}{n} \end{cases}, \quad v_n(x) = \begin{cases} \frac{1}{2} x + \frac{1}{2n}, & x \in \left[-\frac{1}{3n}, \frac{1}{n}\right] \\ |x|, & x \in \left[-1, -\frac{1}{3n}\right] \cup \left[\frac{1}{n}, 1\right] \end{cases}$$

for all $n \in \mathrm{N}$.

(a) It can be directly verified that all the functions u_n and v_n are convex on $[-1, 1]$ for all $n \in \mathrm{N}$.

(b) For the sequence u_n we have $|u_n(x) - |x|| \leqq \frac{2}{n} (x \in [-1, 1]$ and $n \in \mathrm{N})$ and analogously for the sequence v_n we have $|v_n(x) - |x|| \leqq \frac{2}{n} (x \in [-1, 1]$ and $n \in \mathrm{N})$. Hence, both of the sequences u_n and v_n are uniformly convergent to the limit function $f(x) = |x|$ on $[-1, 1]$, i. e. in order to prove the statement (b) of our lemma we can take $f_n = u_n$ or we can take $f_n = v_n$.

(c) If we take $f(x) = |x|$ then obviously we have $f'_+(0) = 1$ and $f'_-(0) = -1$.

On the other hand if we define $f_n = u_n$ then it follows that $f'_{n+}(0) = f'_{n-}(0) = 0$. Hence, the sequences f'_{n+} and f'_{n-} are convergent and the limit point is 0 wherefrom the proof of the statement (c) follows. The same statement can be proved if we consider the sequence $f_n = v_n$.

(d) To prove the statement (d) we have constructed two sequences u_n and v_n which satisfy all of the conditions (a) — (c).

Now we shall consider the sequence f_n defined by $f_{2k}(x) = u_{2k}(x)$, $f_{2k-1}(x) = v_{2k-1}(x)$ for all $x \in [-1, 1]$ and all $k \in \mathrm{N}$. It can be directly verified that $f'_{2k+}(0) = f'_{2k-}(0) = 0$ and $f'_{2k-1+}(0) = f'_{2k-1-}(0) = \frac{1}{2}$. From the above statement (a) it follows that the sequence f_n converges uniformly to the function $f(x) = |x|$ on $[-1, 1]$. Hence, both of the sequences $f'_{n+}(0)$ and $f'_{n-}(0)$ have two points of accumulation 0 and $\frac{1}{2}$ wherefrom the proof of our lemma follows.

Besides the counterexample given in lemma 2.1. we can prove that the sequences of derivatives of convergent sequence of convex functions possess some good properties.

Theorem 2.1. *Let us suppose that the sequence $f_n (n \in \mathrm{N})$ of convex functions (where all the functions are defined on the same domain I) converges point by point on I, to the finite function f. Then:*

(a) *for every point $x_0 \in \operatorname{int} I$ we have*

$$(2.1) \qquad \overline{\lim_{n \to +\infty}} f'_{n+}(x_0) \leqq f'_+(x_0).$$

(b) *Under the above suppositions the sequence $f_n (n \in \mathrm{N})$ and the point $x_0 \in \operatorname{int} I$ can be chosen in such a manner that we have $\overline{\lim_{n \to +\infty}} f'_{n+}(x_0) < f'_+(x_0)$.*

Proof. The statement (b) of our theorem is just proved in the above lemma 2.1. It remains only to prove the statement (a).

Since $x_0 \in \operatorname{int} I$, on the basis of the assumptions of our theorem it follows that all values $f'_+(x_0)$ and $f'_{n+}(x_0)$ are finite.

Suppose on the contrary to (2.1) that there exists $x_0 \in \operatorname{int} I$ and a subsequence $f'_{n_k+}(x_0)$ such that

$$(2.2) \qquad \lim_{k \to +\infty} f_{n_k+}(x_0) = A > f_+(x_0).$$

Further on, let us suppose that $\varepsilon > 0$ is fixed but in such a way that we have

$$(2.3) \qquad A - \varepsilon > f'_+(x_0).$$

In virtue of (2.2) it follows that there is $k_0 \in \mathrm{N}$ such that we have $f'_{n_k+}(x_0) > > A - \varepsilon$ for all $k \geqq k_0$. Since all the functions f_{n_k} are convex on I we get that the inequality

$$(2.4) \qquad \frac{f_{n_k}(y) - f_{n_k}(x_0)}{y - x_0} \geqq f'_{n_k+}(x_0) > A - \varepsilon$$

holds for all $y \in \operatorname{int} I$, and all $k \geqq k_0$, where we have supposed that $y > x_0$. Letting $k \to +\infty$, from (2.4) it follows that the condition

$$(2.5) \qquad \frac{f(y) - f(x_0)}{y - x_0} \geqq A - \varepsilon$$

is satisfied for every $y > x_0$. Let now $y \to x_0 + 0$, wherefrom and by the use of (2.3) and (2.5) it follows that we have $f'_+(x_0) \geqq A - \varepsilon > f'_+(x_0)$ which is a contradiction. From this contradiction it follows that we can conclude that $A \leqq f'_+(x_0)$ which proves that (2.1) holds true.

We have showed in lemma 2.1. that if the sequence f'_{n+} (or the sequence f'_{n-}) converges, this still does not mean that the equality $\lim_{n \to +\infty} f'_{n+}(x) = = f'_+(x)$ (or the equality $\lim_{n \to +\infty} f'_{n-}(x) = f'_-(x)$) holds true. Meanwhile, the question of the validity of that equality, can be solved by using the theorem 2.1.

Theorem 2.2. *Suppose that a sequence of functions $f_n (n \in \mathrm{N})$ convex on I, converges point by point, everywhere in I, to the finite function f. Then the equality*

$$(2.6) \qquad \lim_{n \to +\infty} f'_{n+}(x) = f'_+(x)$$

holds for some point $x \in \operatorname{int} I$, if and only if for the same point we have

$$(2.7) \qquad \underline{\lim}_{n \to +\infty} f'_{n+}(x) = f'_+(x).$$

Proof. It can be directly verified that (2.7) follows from (2.6).

On the contrary, let us suppose that the relation (2.7) holds true. Since the suppositions of theorem 2.2. are identical to those of theorem 2.1. then (2.1) holds true, which besides (2.7) implies that the sequence $f'_{n+}(x)$ converges and that its limit value is $f'_+(x)$, wherefrom the proof of theorem 2.2. follows.

Without proof, we shall give the statements of two theorems which are analogous to that of theorems 2.1 and 2.2 and which are valid for the sequences of left derivatives of a given sequence f_n of convex functions.

Theorem 2.3. *Let us suppose that the sequence $f_n (n \in \mathrm{N})$ of convex functions on I converges point by point everywhere in I to the finite function f. Then:*

(a) *for every point $x_0 \in \operatorname{int} I$ we have*

$$(2.8) \qquad \underline{\lim}_{n \to +\infty} f'_{n-}(x_0) \geqq f'_-(x_0).$$

(b) *Under the above conditions the sequence $f_n (n \in \mathrm{N})$ and the point $x_0 \in \operatorname{int} I$ can be chosen in such a way that*

$$\underline{\lim}_{n \to +\infty} f'_{n-}(x_0) > f'_-(x_0).$$

Theorem 2.4. *Suppose that the sequence $f_n (n \in \mathrm{N})$ of functions convex on I, converges point by point everywhere in I to the finite function f. Then, for some point $x \in \operatorname{int} I$, we have*

$$(2.9) \qquad \lim_{n \to +\infty} f'_{n-}(x) = f'_-(x)$$

if and only if the condition

$$(2.10) \qquad \overline{\lim}_{n \to +\infty} f_{n-}(x) = f_-(x)$$

is satisfied for the same point.

Now we shall return to the sets $G_n (n \in \mathrm{N}_0)$ and G defined above. Let us suppose that the sequence $f_n (n \in \mathrm{N})$ satisfies the conditions of theorem 2.1 on some interval I and suppose that $x_0 \in \operatorname{int} I \setminus \bigcup_{k=1}^{+\infty} G_k$ (i. e. the point x_0 is such that all the functions $f_n (n \in \mathrm{N})$ are differentiable at x_0). Then, as we can conclude from the examples constructed in lemma 2.1, the limit function f (which is convex) does not have to be differentiable at x_0, even in the case when the sequence $f'_n(x_0) (n \in \mathrm{N})$ is convergent. However, in some sense we can prove the opposite statement.

Theorem 2.5. *Let us suppose that the sequence $f_n (n \in \mathrm{N})$ of functions, defined on some interval I, satisfies the conditions of theorem 2.1. Then, for every point $x_0 \in \operatorname{int} I \setminus G_0$ we have*

$$\lim_{n \to +\infty} f'_{n+}(x_0) = \lim_{n \to +\infty} f'_{n-}(x_0) = f'(x_0). \tag{2.11}$$

Proof. For an arbitrary point $x_0 \in \operatorname{int} I$ the inequality $f'_{n-}(x_0) \leqq f'_{n+}(x_0)$ is valid for every $n \in \mathrm{N}$. Therefrom it follows that we have

$$\underline{\lim}_{n \to +\infty} f'_{n-}(x_0) \leqq \underline{\lim}_{n \to +\infty} f'_{n+}(x_0) \leqq \overline{\lim}_{n \to +\infty} f'_{n+}(x_0).$$

By the suppositions of our theorem we have that the conditions (2.1) and (2.8) are valid so that for if $x_0 \in \operatorname{int} I \setminus G_0$ we obtain

$$f'(x_0) = f'_-(x_0) \leqq \underline{\lim}_{n \to +\infty} f'_{n-}(x_0) \leqq \overline{\lim}_{n \to +\infty} f'_{n+}(x_0) = f'(x_0)$$

which proves the theorem i. e. (2.11).

Under the condition of the theorem 2.1 from (2.11) we obtain the statement of the theorem 1.1 or equivalently to that of theorem 1.2 if in addition we assume that $x_0 \in \operatorname{int} I \setminus G$, where G is the above defined set.

From the theorems proved above and theorem 1.2 it can be directly concluded that the following statement is valid: if the suppositions of theorem 2.1 are satisfied for the functions f and $f_n (n \in \mathrm{N})$, on some interval I then:

(i) the sequence $f'_{n+}(x_0)$ (or the sequence $f'_{n-}(x)$) can be divergent in at most countable set of points $x \in \operatorname{int} I$.

(ii) the set of points $x \in \operatorname{int} I$ at which the sequence $f'_{n+}(x)$ (or the sequence $f'_{n-}(x)$) converges but at the same time the inequality (2.6) (or respectively the inequality (2.9)) is not satisfied, is at most countable.

Those results directly follows from (2.1) — (2.10).

3. All the sequences and functions which we have considered in the part 2 of the present paper are such that the values of those functions are finite real numbers, i. e. the points $+\infty$ and $-\infty$ are exluded from our considerations. This was ensured by the supposition that all finite and convex functions are considered in the interior of the corresponding domain of definition of those functions, where the left and right derivatives are finite functions.

In this part of the present paper we shall consider the above defined sequences and functions but at this time our considerations will be concentrated to the end points of the domain of convexity. Hence, it is sufficient to consider the intervals of the form $[a, b)\,(-\infty < a < b \leqq +\infty)$ or $(a, b]$ $(-\infty \leqq a < b < +\infty)$ at which the right or the left derivative of a convex function f respectively satisfies $-\infty \leqq f'_+(a) < +\infty$ or $-\infty < f'_-(b) \leqq +\infty$. In other words we will consider the sequences and functions whose values are possibly $-\infty$ or $+\infty$ at the end points a or b. At the same time we will regard that the limit values, the left and right derivatives of the considered functions, exist in the extended sense.

In the following theorem we will show that the result of theorem 2.1 remains valid even in the case if we consider the left end point of the interval $[a, b)$ (where in proving that theorem we shall respect the above conventions).

Theorem 3.1. *Let us suppose that the sequence of functions f_n $(n \in \mathbf{N})$ convex on $[a, b)$ converges point by point everywhere in $[a, b)$ to the finite functions f. Then we have*

$$\overline{\lim_{n \to +\infty}} f'_{n+}(a) \leq f'_{+}(a). \tag{3.1}$$

Proof. The set $\mathbf{N}$ of all natural numbers we will divide into two subsets $\mathbf{N}_i$ $(i=1, 2)$ in such a manner that

$$\mathbf{N}_1 = \{n \in \mathbf{N} \mid f'_{n+}(a) > -\infty\}, \quad \mathbf{N}_2 = \{n \in \mathbf{N} \mid f'_{n+}(a) = -\infty\}.$$

Hence, we have $\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_2$ and $\mathbf{N}_1 \cap \mathbf{N}_2 = \varnothing$. To prove the relation (3.1) we have to distinguish the following cases:

(i) In the set $\mathbf{N}_1$ there is only finite number of elements.

In that case there exists $n_0 \in \mathbf{N}$ such that we have $n \in \mathbf{N}_2$ for all $n \geq n_0$. Therefrom it follows that we have $\lim_{n \to +\infty} f'_{n+}(a) = -\infty$. In that way in both cases $f'_{+}(a) = -\infty$ or $f'_{+}(a) > -\infty$ we conclude that the relation (3.1) is true.

(ii) In the set $\mathbf{N}_1$ there is an infinite number of elements.

Now, there is a subsequence $f'_{n_k+}(a)$ $(k \in \mathbf{N})$ of the sequence $f'_{n+}(a)$ $(n \in \mathbf{N})$ whose every member is a finite real number (hence whose every member is $> -\infty$). Now, the proof of the relation (3.1) is analogous to that of theorem 2.1 in both of the cases $f'_{+}(a) = -\infty$ or $f'_{+}(a) > -\infty$ where in proving that fact we have to start from the sequence $f'_{n_k+}(a)$ $(k \in \mathbf{N})$.

Without proof we will give the following theorem which holds true for the sequence of the left derivatives and which is analogous to the above theorem.

Theorem 3.2. *Let us suppose that the conditions of the theorem* 3.1 *are satisfied for the functions f and f_n $(n \in \mathbf{N})$ on some interval $(a, b]$. Then we have*

$$\underline{\lim_{n \to +\infty}} f'_{n-}(b) \geq f'_{-}(b). \tag{3.2}$$

By using the relations (3.1) and (3.2) it is possible, as in the part 2 of this paper, to investigate the conditions under which the sequences $f'_{n+}(a)$ or $f'_{n-}(b)$ converge respectively to the limit values $f'_{+}(a)$ or $f'_{-}(b)$. Those theorems will be not stated here explicitly because of their complete analogy with the above theorems.

From the theorems 2.1. and 3.1 it follows that the relation (2.1) holds true for every $x_0 \in [a, b)$ and analogously from the theorems 2.3 and 3.2 we get that the relation (2.8) is valid for all $x_0 \in (a, b]$.

4. In this last part of our paper, we will give some consequences of the theorems proved above and at the same time we will give a few remarks in connection with the possibility of weakening assumptions of previously proved theorems.

As one can expect from the above theorems directly follows an analogous result for infinite series of convex functions.

Theorem 4.1. *Assume that every one of the functions* $f_n (n \in \mathrm{N})$ *is convex on* I *and that the series* $\sum_{k=1}^{+\infty} f_k(x)$ *converges point by point everywhere in* I *to the finite function. Then for every point* $x_0 \in \operatorname{int} I$ *we have*

$$\overline{\lim_{n\to+\infty}} \sum_{k=1}^{n} f'_{k+}(x_0) \leq f'_+(x_0).$$

Besides, the series $\sum_{k=1}^{+\infty} f'_{k+}(x_0)$ *converges and we have the equality*

$$\sum_{k=1}^{+\infty} f'_{k+}(x_0) = f'_+(x_0)$$

if and only if the condition

$$\overline{\lim_{n\to+\infty}} \sum_{k=1}^{n} f'_{k+}(x_0) = f'_+(x_0)$$

is satisfied.

Analogous results can be obtained for the series of the left derivatives of the functions convex on some domain I.

For the infinite products of convex functions we can state theorems analogous to the previous one based on the above theorems. Stating those theorems we have to take into account the conditions which ensure the convergence point by point of the infinite products in consideration (in connection with those conditions see [2] page 20).

In all of the above theorems we have supposed that the considered sequence $f_n (n \in \mathrm{N})$ converges point by point in the interval in question. Well known are the conditions which ensure the convergence, of that type, of the sequences of the convex function (see [2] page 20). For example, in the monograph [2] we can find the following conditions:

(a) A sequence of convex functions should be such that we have:

(i) $f_{n+1}(x) \leq f_n(x) (n \in \mathrm{N})$ for every $x \in I$ and

(ii) there exists $x_0 \in \operatorname{int} I$ such that the sequence $f_n(x_0) (n \in \mathrm{N})$ is bounded from below.

(b) A sequence of convex functions should be such that we have:

(i) $f_{n+1}(x) \geq f_n(x) (n \in \mathrm{N})$ for every $x \in I$ and

(ii) there exist $a, b \in I$ such that the sequences $f_n(a)$ and $f_n(b)$ are bounded from above.

(c) A sequence of convex functions $f_n (n \in \mathrm{N})$ converges to the finite function f in a set which is everywhere dense in the set I.

(d) A sequence $f_n (n \in \mathrm{N})$ of convex functions is bounded point by point in I.

Every one of the conditions (a) — (d) implies the convergence point by point of the sequence f_n in I. Especialy, the condition (c) implies that the sequence $f_n (n \in \mathrm{N})$ converges point by point everywhere in int I.

Hence, everyone, of the conditions (a) — (c) can be taken as the supposition in the above theorems instead of the assumption of convergence point by point.

From the assumption (d) it follows that there exists a subsequence f_{n_k} $(k \in \mathrm{N})$ of the sequence f_n $(n \in \mathrm{N})$ which converges uniformly on every compact subset of the set int I. This time we can derive the corresponding conclusions for the sequences of right and left derivatives of the functions belonging to the subsequence f_{n_k}.

All the statements we have considered do not change their sense if instead of convex functions we consider strictly convex functions.

REFERENCES

1. Masatsugu Tsuji: *On Riesz' fundamental theorem on subharmonic functions*. Tohôku Mathematical Journal, second series. Vol. **4** (1952), 131—140.
2. A. W, Roberts ang D. E. Varberg: *Convex Functions*. New York and London 1973.

Elektrotehnički fakultet
11000 Beograd, Jugoslavija

O PONAŠANJU NIZOVA LEVIH I DESNIH IZVODA KONVERGENTNOG NIZA KONVEKSNIH FUNKCIJA

Ivan B. Lacković

U radu su ispitivani uslovi pod kojima niz levih i desnih izvoda konvergentnog niza konveksnih funkcija konvergira ka izvodnoj funkciji granične funkcije datoga niza.