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721. A WEIGHTED-ARITHMETIC-GEOMETRIC MEANS INEQUALITY*

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We obtain an infinite class of generalizations of the arithmetic-geometric means inequality in which weights less than or equal to 1 are inserted in the arithmetic mean

The arithmetic-geometric means inequality has been generalized in a variety of ways. FINK and JODEIT [1] have shown that for $x_i > 0$

$$(E_n): \qquad \sum_{i=1}^{n} x_i{}^n \prod_{k=1}^{n} q\left(\frac{x_k}{x_i}\right) > n \prod_{k=1}^{n} x_k$$

unless all x_i have the same value, if q is replaced by

$$(1) \qquad q_n(x) = 1 - (1-x)^n \quad 0 \leqq x \leqq 1; \; q_n(x) \equiv 1, \; x > 1.$$

If $q(x) \equiv 1$, then (E_n) is the usual arithmetic-geometric means inequality. Observe that in general, the products in the left hand side of (E_n) are less than 1 unless x_i is the smallest among $x_1, \ldots, x_n$. One wonders what special properties of q_n are necessary for (E_n) to hold. It is our purpose to construct a family of functions for which (E_n) holds and which includes $q(x) \equiv 1$. The FINK-JODEIT result does not contain this special case.

The class of functions considered for the inequality should give equality when all the x_i are the same. This requires that $q(1) = 1$. The other conditions are modeled after some of the properties of q_n.

Let $q \in Q_n$ if

i) $q^{(n+1)}$ exists on (0,1) and is in $L(0,1)$;

ii) $q^{(i)}(1) = 0 \; (i = 1, \ldots, n-1), \; q(1) = 1$;

iii) $\int_0^t x^n (-1)^n q^{(n+1)}(x) \geqq 0, \; 0 \leqq t \leqq 1,$

$$(-1)^n \left[\int_0^1 x^n q^{(n+1)}(x)\, dx - q^{(n)}(1) \right] \geqq 0;$$

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iv) $q(0)\geqq 0$, $q(x)\equiv 1$ for $x\geqq 1$.

Note that $q_m\in Q_n$ if $m\geqq n$, and $q(x)\equiv 1\in Q_n$.

Theorem. *For every $q\in Q_n$, (E_n) holds for $x_i>0$ unless all x_i are the same, in which case equality holds.*

To facilitate the proof we introduce the notation

$$x_+^n=x^n \text{ if } x\geqq 0,\ x_+^n=0 \text{ if } x<0, \tag{2}$$

with the convention that $0^0=1$.

We divide the proof into two parts. Firts assume that $q^{(n+1)}(x)\not\equiv 0$ on $(0,1)$. Then it may be verified in turn that for $0\leqq x\leqq 1$,

$$q^{(n)}(x)=\int_x^1 q^{(n+1)}(t)\,\mathrm{d}t-q^{(n)}(1)=\int_0^1 (t-x)_+^0\mathrm{d}\lambda(t), \tag{3}$$

where $\mathrm{d}\lambda(t)=-q^{(n)}(1)\,\delta_1+q^{(n+1)}(t)\,\mathrm{d}t$ and δ_1 is a point mass at 1. Using the zeros of $q^{(i)}$ at 1 it follows that

$$q'(x)=\int_x^1 (u-x)^{n-2}\,q^{(n)}(u)\,\frac{(-1)^{n-1}}{(n-2)!}\,\mathrm{d}u=\int_0^1 (u-x)_+^{n-2}\,q^{(n)}(u)\,\frac{(-1)^{n-1}}{(n-2)!}\,\mathrm{d}u. \tag{4}$$

Combining (3) and (4),

$$\begin{aligned} q'(x)&=\int_0^1 (u-x)_+^{n-2}\,\mathrm{d}u\int_0^1 (t-u)_+^0\,\mathrm{d}\lambda(t)\,\frac{(-1)^{n-1}}{(n-2)!}\\ &=\int_0^1 \mathrm{d}\lambda(t)\,\frac{(-1)^n}{(n-2)!}\int_0^1 (t-u)_+^0\,(u-x)_+^{n-2}\,\mathrm{d}u\\ &=\int_0^1 \mathrm{d}\lambda(t)\,\frac{(-1)^n}{(n-2)!}\,(t-x)_+^0\int_x^t (u-x)^{n-2}\,\mathrm{d}u\\ &=\int_0^1 (t-x)_+^{n-1}\,\mathrm{d}\lambda(t)\,\frac{(-1)^n}{(n-1)!}. \end{aligned} \tag{5}$$

Finally

$$q(s)=\int_0^s q'(x)\,\mathrm{d}x+q(0)=\int_0^1 (s-x)_+^0\,\mathrm{d}x\int_0^1 (t-x)_+^{n-1}\,\mathrm{d}\lambda(t)\frac{(-1)^n}{(n-1)!}+q(0). \tag{6}$$

Interchanging the order of integration we finally get

$$q(s)=\int_0^1 k(s,t)\,\mathrm{d}\mu(t), \tag{7}$$

where

$$\mathrm{d}\mu(t)=q(0)\,\delta_0+\frac{(-1)^n}{n!}\,\mathrm{d}\lambda(t)$$

and

$$k(s,t)=n\int_0^1 (s-x)_+^0(t-x)_+^{n-1}\,\mathrm{d}x=t^n-(t-s)_+^n \qquad 0\leqq t,\ s\leqq 1 \tag{8}$$

and $k(s,t)\equiv k(1,t)$ when $s\geqq 1$.
The last condition $k(s,t)=k(1,t)$ extends the function q to $[1,\infty)$ so that $q(s)\equiv 1$ there. Now let $x_1,\dots,x_n$ be given positive numbers not all the same. The inequality (E_n) is equivalent to

$$\int_0^1\cdots\int_0^1\left(\sum_{i=1}^n x_i^n\prod_{k=1}^n k\left(\frac{x_k}{x_i},t_k\right)-n\prod_{j=1}^n x_j\prod_{k=1}^n k(1,t_k)\right)\mathrm{d}\mu(t_1)\cdots\mathrm{d}\mu(t_n)>0. \tag{I_n}$$

The integrand is non-negative. To see this, note first that $k(x,0)=0$, and second that if $t_k>0$, then $k(x,t)=q_n(x/t)\,k(1,t)$, see (1). Thus the integrand has a factor $\prod_{k=1}^n k(1,t_k)$ with the other factor being

$$\sum_{i=1}^n x_i^n\prod_{k=1}^n q_n\left(\frac{x_k}{x_i t_k}\right)-n\prod_{j=1}^n x_j\equiv G(t_1,\dots,t_n). \tag{9}$$

Now $q_n(x/t)$ is decreasing in t so that $G(t_1,\dots,t_n)$ is larger than $G(1,\dots,1)$. This quantity is positive by the FINK-JODEIT result. If the measure $\mathrm{d}\mu\geqq 0$, this would complete the proof. However, one can verify that in this case $q(x)\geqq q_n(x)$ and the proof is more direct.

The one dimensional case is a motivation for what we do. If $\int_0^1 fg\geqq 0$ is to be shown and $f(1)\geqq 0$ with $f'\leqq 0$, then we do not need require $g\geqq 0$. An integration by parts shows that if we take $G(x)=\int_0^x g$ then

$$\int_0^1 fg=f(1)\,G(1)+\int_0^1 G(x)\,(-f'(x))\geqq 0 \text{ if only } G(x)\geqq 0$$

on $[0,1]$. This allows g to be negative near 1 if it is sufficiently positive near 0. The multi-dimensional case is analogous but the integration by parts is awkward.

$$\frac{\delta G}{\delta t_j}=\sum_{i=1}^n x_i^n\prod_{k\neq j} q_n\left(\frac{x_k}{x_i t_k}\right)q_n'\left(\frac{x_j}{x_i t_j}\right)\left(\frac{x_j}{x_i}\right)\left(-\frac{1}{(t_j)^2}\right)\leqq 0 \tag{10}$$

and this holds even at $t_k=0$ since $q_n\left(\frac{x_k}{x_i\,t_k}\right)\equiv 1$ if $\frac{x_k}{x_i}\geq t_k$. Thus G decreases in the coordinate directions and has its minimum at (1, 1, ..., 1). This implies that there is a measure ν such that

$$(11)\qquad G(t_1,\ldots,t_n)=\int_{t_1}^{1}\cdots\int_{t_n}^{1}d\nu(s_1,\ldots,s_n)=\int_0^1\cdots\int_0^1\prod_{i=1}^{n}(s_i-t_i)_+^0\,d\nu(s).$$

The outline of the construction of this measure follows the completion of the proof of the theorem. Recalling that $k(1,t)=t^n$ we can rewrite (I_n) as

$$\int_0^1\cdots\int_0^1 G(t_1,\ldots,t_n)\prod_{k=1}^{n}t_k^n\,d\mu(t_1)\cdots d\mu(t_n)\geq 0$$

which using (11) and an interchange of order of integration is equivalent to

$$(12)\qquad \int_0^1\cdots\int_0^1 d\nu(s)\int_0^1\cdots\int_0^1\prod_{i=1}^{n}(s_i-t_i)_+^0\prod_{j=i}^{n}t_j\,d\mu(t_1)\cdots d\mu(t_n')\geq 0,$$

the inner n-fold integral is $\prod_{i=1}^{n}\int_0^s t_i^n\,d\mu(t_i)$.

Thus (12) is a correct inequality if

$$(13)\qquad \int_0^s t^n\,d\mu(t)\geq 0.$$

Since $d\mu=q(0)\,\delta_0+\frac{(-1)^n}{n!}\left[q^{(n+1)}(t)\,dt-q^{(n)}(1)\,\delta_1\right]$ the condition (13) means that

$$(-1)^n\int_0^s t^n q^{(n+1)}(t)\,dt\geq 0\quad 0\leq s<1\ \text{ and }\ (-1)^n\left[\int_0^1 t^n q^{(n+1)}(t)\,dt-q^{(n)}(1)\right]\geq 0.$$

These both follow from condition (iii). Note that this last requirement is necessary since the point (1, ..., 1) has positive ν measure.

To complete the proof of the theorem we look at functions in Q_n with $q^{(n+1)}\equiv 0$. It may be verified that the general solution of this differential equation with the initial conditions ii) is

$$q_a(x)=1-a(1-x)_+^n$$

the conditions (iii) and (ii) that $(-1)^n q^{(n)}(1)\leq 0$ and $q(0)\geq 0$ reguire $0\leq a\leq 1$ so that $q_a(x)\geq q_1(x)$. Thus (E_n) holds since q_1 is in fact what we called q_n and is the FINK-JODEIT result.

Outline of the proof of (11). The two dimensional case will shed some light. It is easy to verify by integration by parts that for any function sufficiently diferentiable

$$G(x, y)=G(1, 1)-\int_x^1 G_1(\xi, 1)\,d\xi-\int_y^1 G_2(1, \eta)\,d\eta + \int_x^1\int_y^1 G_{12}(\xi, \eta)\,d\xi\,d\eta \tag{14}$$

so that if one defines the measure ν by

$$d\nu(x, y)=G(1, 1)\,\delta_{(1,1)}-G_1(x, 1)\,dx\,\delta_1(y)-G_2(1, \eta)\,d\eta\,\delta_1(x) + G_{12}(x, \eta)\,dx\,dy \tag{15}$$

then $G(x, y)=\int_x^1\int_y^1 d\nu(\xi, \eta)$.

If one now replaces $G(x, y)$ by $G(x, y, z)$ in (14) regarding z as a parameter and then writes

$$G(1, 1, z)=G(1, 1, 1)-\int_z^1 G_3(1, 1, \zeta)\,d\zeta,$$

$$G_1(\xi, 1, z)=G_1(\xi, 1, 1)-\int_z^1 G_{13}(\xi, 1, \zeta)\,d\zeta$$

etc. one gets

$$\begin{aligned} G(x, y, z)=G(1, 1, 1)&-\int_x^1 G_1(\xi, 1, 1)\,d\xi-\int_y^1 G_2(1, \eta, 1)\,d\eta \\ &-\int_z^1 G_3(1, 1, \zeta)\,d\zeta+\int_x^1\int_y^1 G_{12}(\xi, \eta, 1)\,d\xi\,d\eta \\ &+\int_x^1\int_z^1 G_{13}(\xi, 1, \zeta)\,d\xi\,d\zeta+\int_y^1\int_z^1 G_{23}(1, \eta, \zeta)\,d\eta\,d\zeta \\ &-\int_x^1\int_y^1\int_z^1 G_{123}(\xi, \eta, \zeta)\,d\xi\,d\eta\,d\zeta. \end{aligned}$$

The pattern for n variables is now clear. For the given G of the theorem, note that each diferentiation brings out a minus sign as long as no variable is differentiated twice. Thus $G_i \leq 0$, $G_{ij} \geq 0$, $G_{ijk} \leq 0$ etc. It follows that (11) holds with a non-negative measure analogous to (15). We offer a class of examples:

Let

$$q(x)=1-(1-x)_+^n-a^2\left[(1-x)_+^{n+1}+\frac{n}{2-n}(1-x)_+^{n+2}+\frac{2}{n-2}(1-x)_+^{n+3}\right],\quad n>2.$$

Then it can be verified that (i), (ii) and (iv) are satisfied with $q(0)=0$. Since $(-1)^n q^{(n)}(1)=-n!$, (iii) is satisfied if $(-1)^n \int_0^x t^n q^{(n+1)}(t)\,dt \geqq 0$, $0 \leqq x \leqq 1$. In fact

$$\frac{(-1)^n}{(n+1)!}\int_0^x t^n q^{(n+1)}(t)\,dt=\frac{a^2x^{n+1}}{n-2}\left[\frac{4n+6}{n+1}-\frac{3}{2}x+(n+2)x^2\right]\geqq 0.$$

Furthermore the [ ] $\geqq 0$ near $x=1$ so that $q \leqq q_n$ near $x=1$. The number a is picked sufficiently small so that $q' \geqq 0$. No such examples exist for $n=2$, since (E_n) is for $x_1>x_2$, the inequality

$$x_1^2\,q\left(\frac{x_2}{x_1}\right)+x_1^2 \geqq 2\,x_1 x_2.$$

Let $t=\frac{x_2}{x_1}$ then for $0<t\leqq 1$ we have $q(t)\geqq 2t-t^2=q_2(t)$. Thus q_2 furnishes the best inequality for (E_2), in fact for q_2, the two sides are equal. Even for $n=3$, it does not appear that there is an easy minimal function for which (E_3) is true.

REFERENCES

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JEDNA NEJEDNAKOST IZMEĐU TEŽINSKE ARITMETIČKE I TEŽINSKE GEOMETRIJSKE SREDINE

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U radu je dobijena jedna klasa nejednakosti između težinske aritmetičke i težinske geometrijske nejednakosti, pri čemu težine koje se pojavljaju u geometrijskoj sredini nisu veće od 1.