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718. A REFINEMENT OF THE MATHIEU INEQUALITY*

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1. Introduction. Let c be a positive number,

$$S=\sum_{n=1}^{\infty}\frac{2n}{(n^2+c^2)^2}.$$

The MATHIEU inequality

(1) $$S<\frac{1}{c^2}$$

was conjectured in 1890 by MATHIEU [7]. (1) was proved in 1952 by BERG [1] and in 1957 by MAKAI [6]. By DIANANDA [4] in 1976, (1) was refined to

(2) $$S<\frac{1}{c^2}-\frac{1}{16\,c^4}+O\left(\frac{1}{c^5}\right)$$

for large c. On the other hand, it was also established in 1952 by EMERSLEBEN [5] that

(3) $$S>\frac{1}{c^2}-\frac{5}{16\,c^4}$$

for integer c. (3) was extended to real c in 1956 by VAN DER CORPUT and HEFLINGER [3].

More and related developments of (1) can be found in MITRINOVIĆ [8].

In this paper, a further refinement of the MATHIEU inequality (1) is presented as follows:

(4) $$S=\frac{1}{c^2}-\frac{1}{6\,c^4}+O\left(\frac{1}{c^6}\right)$$

for large c.

2. Verification of (4). In order to simplify our subsequent argument, we employ the following notation:

(5) $$g(t)=\frac{1}{2f^2(t)},$$

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where

$$f(t)=\frac{1}{(t-1)\,t+c^2}\,. \tag{6}$$

Using (5) nad (6), a straightforward manipulation yields

$$\frac{2n}{(n^2+c^2)^2}-f(n)+f(n+1)+g(n)-g(n+1)=\Phi(n) \tag{7}$$

where

$$\Phi(t)=\frac{2tc^2}{[(t-1)t+c^2]^2[t(t+1)+c^2]^2}+\frac{2t^5}{(t^2+c^2)^2[(t-1)t+c^2]^2[t(t+1)+c^2]^2}\,. \tag{8}$$

Setting

$$\Phi(x)=\sum_{n=1}^{+\infty}\Phi(x+n) \tag{9}$$

it is easy to see from (8) that

$$\Phi''(t)=\sum_{n=1}^{+\infty}\Phi''(t+n)=O\left(\frac{1}{c^6}\right) \tag{10}$$

holds uniformly on $|t|\leqq\frac{1}{2}$ for large c. By a use of the formula (2.30a) given in CARNAHAN, LUTHER and WILKES [2, p. 76], we have

$$\Phi(0)=\int_{-1/2}^{1/2}\Phi(x)\,dx+O\,(\max_{|x|\leqq\frac{1}{2}}|\Phi''(x)|). \tag{11}$$

From (8) — (11) follows

$$\sum_{n=1}^{\infty}\Phi(n)=\int_{-1/2}^{1/2}\sum_{n=1}^{\infty}\Phi(x+n)\,dx+O\left(\frac{1}{c^6}\right) \tag{12}$$

$$=\int_{1/2}^{\infty}\Phi(x)\,dx+O\left(\frac{1}{c^6}\right).$$

A straightforward integration, with some steps of change of variables omitted, yields

$$\int_{1/2}^{\infty}\Phi(x)\,dx=\int_{1/2}^{\infty}\frac{2x\,dx}{(x^2+c^2)^2}-\int_{1/2}^{\infty}f(x)\,dx+\int_{1/2}^{\infty}f(x+1)\,dx \tag{13}$$

$$+\int_{1/2}^{\infty} g(x)\,\mathrm{d}x-\int_{1/2}^{\infty} g(x+1)\,\mathrm{d}x$$

$$=\int_{1/2}^{\infty}\frac{2\,x\,\mathrm{d}x}{(x^2+c^2)^2}-\int_0^1\frac{\mathrm{d}x}{x^2+c^2-\frac{1}{4}}+\frac{1}{2}\int_0^1\frac{\mathrm{d}x}{\left(x^2+c^2-\frac{1}{4}\right)^2}$$

$$=\left(c^2+\frac{1}{4}\right)^{-1}-\left(c^2-\frac{1}{4}\right)^{-1/2}\operatorname{arctg}\left(c^2-\frac{1}{4}\right)^{-1/2}$$

$$+\frac{1}{4}\left(c^2-\frac{1}{4}\right)^{-1}\left(c^2+\frac{3}{4}\right)^{-1}+\frac{1}{4}\left(c^2-\frac{1}{4}\right)^{-3/2}\operatorname{arctg}\left(c^2-\frac{1}{4}\right)^{-1/2}$$

$$=\frac{1}{3}\frac{1}{c^4}+O\left(\frac{1}{c^6}\right).$$

On the other hand, from (7) follows

$$\sum_{n=1}^{\infty}\Phi(n)=\sum_{n=1}^{\infty}\frac{(2n}{(n^2+c^2)^2}-\frac{1}{c^2}+\frac{1}{2}\frac{1}{c^4}. \tag{14}$$

It is now clear that (4) follows from (12), (13) and (14).

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RAFINIRANJE JEDNE NEJEDNAKOSTI MATHIEU

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Polazeći od rezultata iznetih u monografiji [8] u vezi sa nejednakošću (1) autor daje izvesna poboljšanja iste, koristeći se asimptotskom relacijom (6) koja važi za velike vrednosti c.