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716. ON A BINOMIAL FUNCTIONAL EQUATION AND SOME RELATED EQUATIONS*

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The main object of this paper is to show that in some cases the functional equation (2.1) can (and how it can) be reduced to the binomial functional equation (1.1) or (1.2). Though the procedure for obtaining the general solution of (2.1) is well known, in practice it is more convenient to go via the equation (1.1) or (1.2).

0. Throughout this paper S will denote a nonempty set and $\mathbf{C}$ the set of all complex numbers. For positive integers $n_1, \dots, n_k$, $(n_1, \dots, n_k)$ denotes the highest common factor and $[n_1, \dots, n_k]$ the lowest common multiple of those numbers.

1. Let $g: S \to S$ be a given mapping such that $g^n(x) = x\,(x \in S)$. Consider the functional equation

$$f(x) = af(gx) \tag{1.1}$$

where $a \in \mathbf{C}$ is given, $f: S \to \mathbf{C}$ is unknown function, and gx denotes $g(x)$. The equation (1.1) will be called the *binomial functional equation.*

Theorem 1. *If* $a^n = 1$, *the general solution of the equation* (1.1) *is given by*

$$f(x) = \sum_{\nu=1}^{n} a^\nu F(g^\nu x),$$

where $F: S \to \mathbf{C}$ *is arbitrary. If* $a^n \neq 1$, *the only solution of the equation* (1.1) *is the trivial solution.*

Let g, a and f be as in Theorem 1, and consider the equation

$$f(x) = af(g^p x) \qquad (p \in \mathbf{N},\ p < n). \tag{1.2}$$

Theorem 2. *Let* $m = (n, p)$. *If* $a^{n/m} = 1$, *the general solution of the equation* (1.2) *is given by*

$$f(x) = \sum_{\nu=1}^{n/m} a^\nu F(g^{\nu p} x),$$

where $F: S \to \mathbf{C}$ *is arbitrary. If* $a^{n/m} \neq 1$, *the only solution of* (1.2) *is the trivial solution.*

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Let $g_1, \ldots, g_k: S \to S$, so that $g_i^{n_i}(x) = x$ $(i = 1, \ldots, k)$. Let $f: S^k \to \mathbf{C}$, $a \in \mathbf{C}$, and consider the equation

$$(1.3) \qquad f(x_1, \ldots, x_k) = af(g_1 x_1, \ldots, g_k x_k).$$

Theorem 3. *Let $N = [n_1, \ldots, n_k]$. If $a^N = 1$, the general solution of the equation* (1.3) *is given by*

$$f(x_1, \ldots, x_k) = \sum_{\nu=1}^{N} a^\nu F(g_1{}^\nu x_1, \ldots, g_k{}^\nu x_k),$$

where $F: S^k \to \mathbf{C}$ is arbitrary. If $a^N \neq 1$, the general solution of (1.3) *is*

$$f(x_1, \ldots, x_k) = 0.$$

Let $g_1, \ldots, g_k$, f and a be as in Theorem 3, and consider the equation

$$(1.4) \qquad f(x_1, \ldots, x_k) = af(g_1{}^{p_1} x_1, \ldots, g_k{}^{p_k} x_k)$$

where $p_i \in \mathbf{N}$, $p_i < n_i$ for $i = 1, \ldots, k$.

Theorem 4. *Let $M = \left[\frac{n_1}{(n_1, p_1)}, \ldots, \frac{n_k}{(n_k, p_k)}\right]$. If $a^M = 1$, the general solution of the equation* (1.4) *is given by*

$$f(x_1, \ldots, x_k) = \sum_{\nu=1}^{M} a^\nu F(g_1{}^{p_1 \nu} x_1, \ldots, g_k{}^{p_k \nu} x_k),$$

where $F: S^k \to \mathbf{C}$ is arbitrary. If $a^M \neq 1$, the general solution of (1.4) *is*

$$f(x_1, \ldots, x_k) = 0.$$

Theorems 1, 2, 3 and 4 were proved in [**1**]. In fact, the equations (1.2), (1.3) and (1.4) can be reduced to the binomial functional equation (1.1). We now give some examples.

Example 1.1. Let $g^5 x = x$. For the equation

$$f(x) = af(g^3 x)$$

we have $m = (5, 3) = 1$, and hence if $a^5 = 1$, the general solution of that equation is given by

$$f(x) = \sum_{\nu=1}^{5} a^\nu F(g^{3\nu} x),$$

i.e.

$$f(x) = F(x) + aF(g^3 x) + a^2 F(gx) + a^3 F(g^4 x) + a^4 F(g^2 x),$$

where $F: S \to \mathbf{C}$ is an arbitrary function.

Example 1.2. Let $g^9 x = x$. For the equation

$$f(x) = af(g^6 x)$$

we have $m = (9, 6) = 3$, and hence if $a^3 = 1$, the general solution of this equation is given by

$$f(x) = \sum_{\nu=1}^{3} a^\nu F(g^{6\nu} x),$$

i.e.

$$f(x) = F(x) + aF(g^6 x) + a^2 F(g^3 x),$$

where $F: S \to \mathbf{C}$ is an arbitrary function.

EXAMPLE 1.3. Let $g_1^7x=g_2^7x=x$ and consider the equation

$$f(x, y)=af(g_1^3x,\ g_2^4y). \tag{1.5}$$

We have $M=\left[\frac{7}{(7,3)}, \frac{7}{(7,4)}\right]=7$, and hence, if $a^7=1$ the general solution of (1.5) is given by

$$f(x, y)=\sum_{\nu=1}^{7} a^\nu F(g_1^{3\nu}x, g_2^{4\nu}y),$$

i.e.

$$f(x, y)=F(x, y)+aF(g_1^3x, g_2^4y)+a^2F(g_1^6x, g_2y)+a^3F(g_1^2x, g_2^5y)$$
$$+a^4F(g_1^5x, g_2^2y)+a^5F(g_1x, g_2^6y)+a^6F(g_1^4x, g_2^3y),$$

where $F: S\to \mathbf{C}$ is an arbitrary function.

EXAMPLE 1.4. Let $g_1^{12}x=g_2^{12}x=x$, and consider the equation

$$f(x, y)=af(g_1^6x, g_2^9y). \tag{1.6}$$

In this case $M=\left[\frac{12}{(12,6)}, \frac{12}{(12,9)}\right]=4$, and hence if $a^4=1$, the general solution of (1.6) is given by

$$f(x, y)=\sum_{\nu=1}^{4} a^\nu F(g_1^{6\nu}x, g_2^{9\nu}y),$$

i.e.

$$f(x, y)=F(x, y)+aF(g_1^6x, g_2^9y)+a^2F(x, g_2^6y)+a^3F(g_1^6x, g_2^3y),$$

where $F: S\to \mathbf{C}$ is an arbitrary function.

EXAMPLE 1.5. Let $g_1^2x=g_2^3x=x$, and consider the equation

$$f(x, y)=af(g_1x, g_2^2y). \tag{1.7}$$

In this case $M=\left[\frac{2}{(2,1)}, \frac{3}{(3,2)}\right]=6$, and if $a^6=1$, the general solution of (1.7) is given by

$$f(x, y)=\sum_{\nu=1}^{6} a^\nu F(g_1^\nu x, g_2^{2\nu}y),$$

i.e.

$$f(x, y)=F(x, y)+aF(g_1x, g_2^2y)+a^2F(x, g_2y)+a^3F(g_1x, y)$$
$$+a^4F(x, g_2^2y)+a^5F(g_1x, g_2y),$$

where $F: S\to \mathbf{C}$ is an arbitrary function.

2. In his monograph on functional equations [2] GHERMĂNESCU treated the equations (1.1) and (1.2) in an entirely different manner. In fact, starting from the general results which hold for the equation

$$a_0f(x)+a_1f(gx)+\cdots+a_{n-1}f(g^{n-1}x)=0 \tag{2.1}$$

where $g^n x = x$, $a_k \in \mathbf{C}$, he drew conclusions for the special equations (1.1) and (1.2). Ghermeănscu did not consider the case when the unknown function depends on several variables, nor the case when there are more than one given functions $g_1, \ldots, g_k$.

The equation (2.1) was the subject of many papers; see, for instance, the monograph [3] and the literature given there.

In this section we will show how the general equation (2.1) can in certain cases be reduced to the equation (1.1), or (1.2).

Starting with (2.1) and replacing x by $gx, \ldots, g^{n-1}x$, we arrive at the system

$$
\tag{2.2}
\begin{aligned}
a_0 f(x) + a_1 f(gx) + \cdots + a_{n-1} f(g^{n-1}x) &= 0,\\
a_{n-1} f(x) + a_0 f(gx) + \cdots + a_{n-2} f(g^{n-1}x) &= 0,\\
&\vdots\\
a_1 f(x) + a_2 f(gx) + \cdots + a_0 f(g^{n-1}x) &= 0,
\end{aligned}
$$

which is equivalent to the equation (2.1).

The system (2.2) will have nontrivial solutions if and only if

$$
\det \mathbf{A} = \det \left\| \begin{matrix} a_0 & a_1 & a_2 \cdots a_{n-1} \\ a_{n-1} & a_0 & a_1 \quad a_{n-2} \\ \vdots & & \\ a_2 & a_3 & a_4 \quad a_1 \\ a_1 & a_2 & a_3 \quad a_0 \end{matrix} \right\| = 0,
$$

i.e. if and only if $\operatorname{rank} \mathbf{A} \leqq n-1$.

(i) Assume that $\operatorname{rank} \mathbf{A} = n-1$. Then, since $\mathbf{A}$ is a cyclic matrix, according to a theorem of Segre [4], all submatrices of order $n-1$ are regular.

The system (2.2) is equivalent to the system (S) obtained from it by omitting the last equation. Eliminating $f(g^2x), f(g^3x), \ldots, f(g^{n-1}x)$ between the $n-1$ independent equations of (S), we get

$$
\tag{2.3}
\begin{vmatrix} a_0 & a_2 \cdots a_{n-1} \\ a_{n-1} & a_1 \quad a_{n-2} \\ \vdots & \\ a_2 & a_4 \quad a_1 \end{vmatrix} f(x) + \begin{vmatrix} a_1 & a_2 \cdots a_{n-1} \\ a_0 & a_1 \quad a_{n-2} \\ \vdots & \\ a_3 & a_4 \quad a_1 \end{vmatrix} f(gx) = 0,
$$

which means that the equation (2.1) is, in this case, reduced to the binomial equation (2.3). Moreover, since $\mathbf{A}$ is a cyclic matrix, we have $\det \mathbf{A} = E(\alpha_1) E(\alpha_2) \cdots E(\alpha_n)$, where

$$
E(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1},
$$

and where $\alpha_1, \ldots, \alpha_n$ are roots of the equation $x^n - 1 = 0$. Hence, if α is a root of the equation $x^n - 1 = 0$, we have

$$
\begin{aligned}
a_1 \alpha + a_2 \alpha^2 + \cdots + a_{n-1} \alpha^{n-1} &= -a_0,\\
a_0 \alpha + a_1 \alpha^2 + \cdots + a_{n-2} \alpha^{n-1} &= -a_{n-1},\\
&\vdots\\
a_3 \alpha + a_4 \alpha^2 + \cdots + a_1 \alpha^{n-1} &= -a_2,
\end{aligned}
$$

which implies that (2.3) reduces to

$$f(x)=af(gx)$$

with $a^n=1$. Every solution of the obtained binomial equation is a solution of (2.1).

EXAMPLE 2.1. For the equation

$$f(x)+2f(gx)+f(g^2x)-4f(g^3x)=0 \qquad (g^4x=x)$$

we have rank $\mathbf{A}=3$. Hence, applying the procedure described above, we arrive at the equation

$$f(x)=f(gx),$$

with the general solution

$$f(x)=F(x)+F(gx)+F(g^2x)+F(g^3x),$$

where F is an arbitrary function which is, at the same time, the general solution of the considered equation.

(ii) If rank $\mathbf{A}=1$, then the equation (2.1) is necessarily of the form

$$f(x)+\alpha f(gx)+\alpha^2 f(g^2x)+\cdots+\alpha^{n-1}f(g^{n-1}x)=0, \tag{2.4}$$

where $\alpha^n=1$, and it cannot be reduced to the binomial equation.

However, the general solution of (2.1) is easily constructed. It is given by

$$f(x)=(n-1)F(x)-\alpha F(gx)-\cdots-\alpha^{n-1}F(g^{n-1}x),$$

where F is an arbitrary function.

(iii) Suppose that rank $\mathbf{A}=r$, where $1<r<n-1$. Then there are r independent equations in the system (2.2). Denote by (S_r) the system obtained from (2.2) by omitting the $n-r$ dependent equations. In general, it is possible to eliminate $f(g^{n-r+1}x),\ldots,f(g^{n-1}x)$ between the r equations of the system (S_r) which will lead to an equation of the form

$$A_0f(x)+A_1f(gx)+\cdots+A_{n-r}f(g^{n-r}x)=0. \tag{2.5}$$

It may happen that the equation (2.5) becomes a binomial equation.

EXAMPLE 2.2. For the functional equation

$$af(x)+bf(gx)-af(g^2x)-bf(g^3x)=0 \qquad (a^2+b^2\neq 0)$$

where $g^4x=x$, the rank of the corresponding matrix $\mathbf{A}$ is 2. Nevertheless, elimination of $f(g^3x)$ from the system of two independent equations

$$af(x)+bf(gx)-af(g^2x)-bf(g^3x)=0,$$
$$-bf(x)+af(gx)+bf(g^2x)-af(g^3x)=0$$

leads to the equation $f(x)=f(g^2x)$, with the general solution $f(x)=F(x)+F(g^2x)$, where F is an arbitrary function.

3. Suppose now that $g_k\colon S\to S$ $(k=0,1,\ldots,n-1)$ so that $\{g_0,g_1,\ldots,g_{n-1}\}$ is a group of order n $(g_0x=x)$, and consider the equation

$$a_0f(x)+a_1f(g_1x)+\cdots+a_{n-1}f(g_{n-1}x)=0, \tag{3.1}$$

where $a_k \in \mathbf{C}$ are given, and $f: S \to \mathbf{C}$ is the unknown function. The method explained in section 2 of this paper can also be applied to the equation (3.1) which will, in certain cases, be reduced to a binomial functional equation. However, the general solution of (3.1) will not, in general, coincide with the general solution of the obtained binomial equation, but it will be a subset of it.

Without going into details, we shall briefly consider a couple of examples.

EXAMPLE 3.1. The mappings $g_0, g_1, g_2, g_3: \mathbf{C} \to \mathbf{C}$, defined by

$$g_0 x = x, \quad g_1 x = -x \quad g_2 x = \frac{1}{x}, \quad g_3 = -\frac{1}{x},$$

form a group of order 4. Consider the equation

$$af(x) + bf(-x) - af\left(\frac{1}{x}\right) - bf\left(-\frac{1}{x}\right) = 0 \quad (a, b \in \mathbf{C};\ a^2 - b^2 \neq 0). \tag{3.2}$$

Replacing x by $-x$, $\frac{1}{x}$ and $-\frac{1}{x}$, we arrive at the system

$$af(x) + bf(-x) - af\left(\frac{1}{x}\right) - bf\left(-\frac{1}{x}\right) = 0,$$

$$bf(x) + af(-x) - bf\left(\frac{1}{x}\right) - af\left(-\frac{1}{x}\right) = 0,$$

$$-af(x) - bf(-x) + af\left(\frac{1}{x}\right) + bf\left(-\frac{1}{x}\right) = 0,$$

$$-bf(x) - af(-x) + bf\left(\frac{1}{x}\right) + af\left(-\frac{1}{x}\right) = 0,$$

which is equivalent to (3.2).

Eliminating $f\left(-\frac{1}{x}\right)$ between the first two equations of this system (the other two being their consequences), we find

$$f(x) = f\left(\frac{1}{x}\right). \tag{3.3}$$

The general solution of (3.3), namely

$$f(x) = F(x) + F\left(\frac{1}{x}\right) \quad (F \text{ arbitrary}) \tag{3.4}$$

also satisfies (3.2) and hence it is the general solution of the equation (3.2).

REMARK 3.1. It is easily verified that (3.4) satisfies (3.2) in the case when $a^2 = b^2$, but it is not the general solution of that equation.

EXAMPLE 3.2. Consider the equation

$$f(x) + 2f(-x) + f\left(\frac{1}{x}\right) - 4f\left(-\frac{1}{x}\right) = 0. \tag{3.5}$$

The corresponding equivalent system reads

$$f(x)+2f(-x)+f\left(\frac{1}{x}\right)-4f\left(-\frac{1}{x}\right)=0,$$

$$2f(x)+f(-x)-4f\left(\frac{1}{x}\right)+f\left(-\frac{1}{x}\right)=0,$$

$$f(x)-4f(-x)+f\left(\frac{1}{x}\right)+2f\left(-\frac{1}{x}\right)=0,$$

$$-4f(x)+f(-x)+2f\left(\frac{1}{x}\right)+f\left(-\frac{1}{x}\right)=0.$$

Eliminating $f\left(-\frac{1}{x}\right)$ between the first three equation (the fourth is clearly a consequence of the first three) we obtain the binomial equation

$$f(x)=f(-x). \tag{3.6}$$

The general solution of (3.6) is

$$f(x)=F(x)+F(-x) \qquad (F \text{ arbitrary}) \tag{3.7}$$

but in general (3.7) is not a solution of (3.5). Indeed, substituting (3.7) into (3.5) we find

$$F(x)+F(-x)-F\left(\frac{1}{x}\right)-F\left(-\frac{1}{x}\right)=0, \tag{3.8}$$

which is an equation of type (3.2) with $a=b$.

Hence (see Remark 3.1) a solution of (3.8) is

$$F(x)=G(x)+G\left(\frac{1}{x}\right) \qquad (G \text{ arbitrary})$$

which, combined with (3.7) gives the following solution of (3.5):

$$f(x)=G(x)+G(-x)+G\left(\frac{1}{x}\right)+G\left(-\frac{1}{x}\right) \qquad (G \text{ arbitrary}). \tag{3.9}$$

REMARK 3.2. Functional equations of the form (3.1), where a_k are functional coefficients, were treated by PREŠIĆ [5, 6] (see also [3, Theorem 13.4]). Using PREŠIĆ's results it can be proved that (3.9) is the general solution of the equation (3.5).

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O JEDNOJ BINOMNOJ FUNKCIONALNOJ JEDNAČINI I NEKIM JEDNAČINAMA KOJE SU SA NJOM U VEZI

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U radu se, između ostalog, pokazuje kako se u određenim slučajevima funkcionalna jednačina (2.1) može svesti na binomnu jednačinu (1.1) ili (1.2). Mada postoji postupak za određivanje opšteg rešenja jednačine (2.1), u praksi je podesnije rešavati tu jednačinu preko jednačine (1.1) ili (1.2).