UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 678—№ 715 (1980), 191—198.

713. SOME INEQUALITIES FOR CONVEX FUNCTIONS

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1. P. M. VASIĆ [1] has proved the following:

Theorem A. *If $F_1, \ldots, F_n$ are integrable and convex functions on $[a, b]$ such that $F_k(x) \geqq 0$ for $x \in [a, b]$ and if $F_k(a) = 0$ $(k = 1, \ldots, n)$ and $P(x)$ is a positive and integrable function on $[a, b]$, we have*

$$\left(\int_a^b P(x)\,\mathrm{d}x\right)^{n-1} \left(\int_a^b P(x) F_1(x) \cdots F_n(x)\,\mathrm{d}x\right) \geqq M \left(\int_a^b P(x) F_1(x)\,\mathrm{d}x\right) \cdots \left(\int_a^b P(x) F_n(x)\,\mathrm{d}x\right) \tag{1.1}$$

where

$$M = \frac{\int_a^b P(x)(x-a)^n\,\mathrm{d}x \left(\int_a^b P(x)\,\mathrm{d}x\right)^{n-1}}{\left(\int_a^b P(x)(x-a)\,\mathrm{d}x\right)^n}. \tag{1.2}$$

Equality in (1.1) *holds if and only if $F_k(x) = C_k(x-a)$ $(k = 1, \ldots, n)$.*

Suppose that the functions $f_1, \dfrac{f_2}{\Phi_1}, \ldots, \dfrac{f_n}{\Phi_{n-1}}$ are nonnegative, convex and integrable on $[a, b]$, such that $f_k(a) = 0$ $(k = 1, \ldots, n)$ and $\Phi_i(x)$ $(i = 1, \ldots, n-1)$, decrease.

Introducing in (1.1)

$$F_1(x) = f_1(x), \quad F_k(x) = \frac{f_k(x)}{\Phi_{k-1}(x)} \quad (k = 2, \ldots, n), \quad P(x) = p(x)$$

* Presented October 1, 1980 by A. LUPAŞ.

we obtain

$$(1.1') \qquad \Big(\int_a^b p(x)\,\mathrm{d}x\Big)^{n-1} \int_a^b p(x)\,\frac{f_1(x)\cdots f_n(x)}{\Phi_1(x)\cdots\Phi_{n-1}(x)}\,\mathrm{d}x$$

$$\geqq M\int_a^b p(x) f_1(x)\,\mathrm{d}x \int_a^b p(x)\,\frac{f_2(x)}{\Phi_1(x)}\,\mathrm{d}x \cdots \int_a^b p(x)\,\frac{f_n(x)}{\Phi_{n-1}(x)}\,\mathrm{d}x.$$

The following TCHEBYSHEV's inequality

$$(1.3) \qquad \int_a^b P(x)\,\mathrm{d}x \int_a^b P(x)F(x)G(x)\,\mathrm{d}x \leqq \int_a^b P(x)F(x)\,\mathrm{d}x \int_a^b P(x)G(x)\,\mathrm{d}x$$

holds if F and G are one increasing and the other decreasing function.

Substituting in (1.3)

$$P(x)=p(x),\ F(x)=\frac{f_k(x)}{\Phi_{k-1}(x)},\ G(x)=\Phi_{k-1}(x) \qquad (k=2,\ \ldots,\ n)$$

we obtain

$$\int_a^b p(x)\,\mathrm{d}x \int_a^b p(x) f_k(x)\,\mathrm{d}x \leqq \int_a^b p(x)\,\frac{f_k(x)}{\Phi_{k-1}(x)}\,\mathrm{d}x \int_a^b p(x)\,\Phi_{k-1}(x)\,\mathrm{d}x$$

or

$$(1.3') \qquad \int_a^b p(x)\,\frac{f_k(x)}{\Phi_{k-1}(x)}\,\mathrm{d}x \geqq \int_a^b p(x)\,\mathrm{d}x\,\frac{\int_a^b p(x) f_k(x)\,\mathrm{d}x}{\int_a^b p(x)\,\Phi_{k-1}(x)\,\mathrm{d}x} \qquad (k=2,\ \ldots,\ n).$$

Combining inequalities (1.1′) and (1.3′) we get

$$(1.4) \qquad \int_a^b p(x)\,\frac{f_1(x)\cdots f_n(x)}{\Phi_1(x)\cdots\Phi_{n-1}(x)}\,\mathrm{d}x \geqq M\,\frac{\int_a^b p(x) f_1(x)\,\mathrm{d}x \cdots \int_a^b p(x) f_n(x)\,\mathrm{d}x}{\int_a^b p(x)\,\Phi_1(x)\,\mathrm{d}x \cdots \int_a^b p(x)\,\Phi_{n-1}(x)\,\mathrm{d}x}.$$

Equality in (1.4) holds if and only if $f_k(x)=C_k(x-a)$ $(k=1,\ \ldots,\ n)$ and $\Phi_k(x)=C$ $(k=1,\ \ldots,\ n-1)$.

Th refore we have:

Theorem 1. *If* $f_1,\ \ldots,\ f_n$ *and* $\Phi_1,\ \ldots,\ \Phi_{n-1}$ *are integrable functions,* f_1, $\frac{f_k(x)}{\Phi_{k-1}(x)}$ $(k=2,\ \ldots,\ n)$ *convex functions,* $\Phi_k(x)$ $(k=1,\ \ldots,\ n-1)$ *decrease on* $[a,\ b]$, *so that* $f_k(x)>0$, $x\in(a, b]$, $f_k(a)=0$ ***and*** $\Phi_k(x)>0$ ***on*** $[a, b]$, $p(x)$ *is a positive and integrable function on* $[a, b]$, *then* ***inequality*** (1.4) ***holds*** *where* M *is given by* (1.2).

Since $M\geqq 1$, we see that (1.4) is a sharper inequality than the inequality from paper [**2**].

Note 1. Taking all the Φ_k $(k=1, \ldots, n-1)$ to be constant in (1.4), we have inequality (1.1).

2. For the sake of brevity, we shall say that two sets are of the same sense, when they both increase or both decrease, and of opposite senses, when one increases and the other decreases.

If X_i and Y_i are two sequences of positive numbers, and let (P_i) be another sequence of positive numbers, then

$$(2.1)\qquad \sum_{i=1}^{n} P_i \sum_{i=1}^{n} P_i X_i Y_i \geqq \text{ or } \leqq \sum_{i=1}^{n} P_i X_i \sum_{i=1}^{n} P_i Y_i$$

as X_i and Y_i are of the same sense or opposite senses.

Introducing in (2.1)

$$P_i=p_i x_{im},\ X_i=\frac{x_{i1}\cdots x_{i,m-1}}{y_{i1}\cdots y_{i,m-1}},\quad Y_i=\frac{y_{i,m-1}}{x_{im}}$$

we obtain

$$(2.2)\qquad \sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{im}}{y_{i1}\cdots y_{i,m-1}} \leqq \sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{i,m-1}}{y_{i1}\cdots y_{i,m-2}} \frac{\sum_{i=1}^{n} p_i x_{im}}{\sum_{i=1}^{n} p_i y_{i,m-1}}$$

as $\dfrac{x_{i1}\cdots x_{i,m-1}}{y_{i1}\cdots y_{i,m-1}}$ and $\dfrac{y_{i,m-1}}{x_{im}}$ are of the same sense.

Putting in (2.1)

$$P_i=p_i x_{i,m-1},\ X_i=\frac{x_{i1}\cdots x_{i,m-2}}{y_{i1}\cdots y_{i,m-2}},\ Y_i=\frac{y_{i,m-2}}{x_{i,m-1}},$$

we get

$$(2.3)\qquad \sum_{i=1}^{n} \frac{x_{i1}\cdots x_{i,m-1}}{y_{i1}\cdots y_{i,m-2}} \leqq \sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{i,m-2}}{y_{i1}\cdots y_{i,m-3}} \frac{\sum_{i=1}^{n} p_i x_{i,m-1}}{\sum_{i=1}^{n} p_i y_{i,m-2}}$$

as $\dfrac{x_{i1}\cdots x_{i,m-2}}{y_{i1}\cdots y_{i,m-2}}$ and $\dfrac{y_{i,m-2}}{x_{i,m-1}}$ are of the same sense.

Combining inequalities (2.2) and (2.3) we get

$$\sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{im}}{y_{i1}\cdots y_{i,m-1}} \leqq \sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{i,m-2}}{y_{i1}\cdots y_{i,m-3}} \frac{\sum_{i=1}^{n} p_i x_{i,m-1}}{\sum_{i=1}^{n} p_i y_{i,m-2}} \frac{\sum_{i=1}^{n} p_i x_{im}}{\sum_{i=1}^{n} p_i y_{i,m-1}}. \tag{2.4}$$

If we proceed this procedure introducing in inequality (2.1) substitutions

$$p_i = p_i x_{ij},\ X_i = \frac{x_{i1}\cdots x_{i,j-1}}{y_{i1}\cdots y_{i,j-1}},\ Y_i = \frac{y_{i,j-1}}{x_{ij}}$$

respectively, for $j=m,\ m-1,\ \ldots,\ 3,\ 2$ and inequality (2.4) we obtain

$$\sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{im}}{y_{i1}\cdots y_{i,m-1}} \leqq \frac{\sum_{i=1}^{n} p_i x_{i1}\cdots \sum_{i=1}^{n} p_i x_{im}}{\sum_{i=1}^{n} p_i y_{i1}\cdots \sum_{i=1}^{n} p_i y_{i,m-1}} \tag{2.5}$$

if positive sequences

$$\frac{x_{i1}\cdots x_{i,j-1}}{y_{i1}\cdots y_{i,j-1}} \quad \text{and} \quad \frac{y_{i,j-1}}{x_{ij}} \ (j=2,\ \ldots,\ m) \tag{2.6}$$

are both of the same sense. In the case of opposite sense we obtain the opposite inequality. This proves the following theorem.

Theorem 2.1. *Let positive sequences* $x_{i1},\ \ldots,\ x_{im};\ y_{i1},\ \ldots,\ y_{i,m-1}$ *and* $p_i (i=1,\ \ldots,\ n)$, *be given. Then the inequality* (2.5) *holds under the condition* (2.6).

Equality in (2.5) *holds if sequences* $x_{i1},\ \ldots,\ x_{im}$ *and* $y_{i1},\ \ldots,\ y_{i,m-1}$ *are constant or if*

$$\frac{x_{i1}}{y_{i1}} = \frac{x_{i2}}{y_{i2}} = \cdots = \frac{x_{i,m-1}}{y_{i,m-1}}.$$

Note 1. If we substitute m with $n+m$ and *if* $y_{im} = \cdots = y_{i,m+n-1} = C\ (C=\text{const},\ i=1,\ \ldots,\ n)$, we obtain

$$\sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{im}\cdots x_{i,m+n}}{y_{i1}\cdots y_{i,m-1}} \leqq \frac{1}{\left(\sum_{i=1}^{n} p_i\right)^n} \frac{\sum_{i=1}^{n} p_i x_{i1}\cdots \sum_{i=1}^{n} p_i x_{i,n+m}}{\sum_{i=1}^{n} p_i y_{i1}\cdots \sum_{i=1}^{n} p_i y_{i,m-1}}. \tag{2.7}$$

It can be proved similarly that if $x_{im} = \cdots = x_{i,m+n} = C\ (C=\text{const},\ i=1,\ \ldots,\ n)$, then

$$\sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{i,m-1}}{y_{i1}\cdots y_{i,m+n}} \leqq \left(\sum_{i=1}^{n} p_i\right)^{n+1} \frac{\sum_{i=1}^{n} p_i x_{i1}\cdots \sum_{i=1}^{n} p_i x_{i,n-1}}{\sum_{i=1}^{n} p_i y_{i1}\cdots \sum_{i=1}^{n} p_i y_{i,n+m}}. \tag{2.8}$$

If $x_{im}=C$ ($C=$const), we obtain

$$\sum_{i=1}^{n} p_i \frac{x_{i1}\cdots x_{i,m-1}}{y_{i1}\cdots y_{i,n-1}} \leq \sum_{i=1}^{n} p_i \frac{\sum_{i=1}^{n} p_i x_{i1}\cdots \sum_{i=1}^{n} p_i x_{i,m-1}}{\sum_{i=1}^{n} p_i y_{i1}\cdots \sum_{i=1}^{n} p_i y_{i,m-1}}. \tag{2.9}$$

Note 2. If we put $y_{ij}=C$ ($C=$const), ($j=1, \dots, n-1$) then from inequality (2.5) it follows

$$\sum_{i=1}^{n} p_i x_{i1}\cdots x_{im} \geq \frac{1}{\left(\sum_{i=1}^{n} p_i\right)^{m-1}} \sum_{i=1}^{n} p_i x_{i1}\cdots \sum_{i=1}^{n} p_i x_{im}. \tag{2.10}$$

The last inequality generalized TCHEBYSHEV's inequality.

If we suppose that $x_{i1}=\cdots=x_{im}=x_i$, we obtain

$$\sum_{i=1}^{n} p_i x_i^m \geq \frac{\left(\sum_{i=1}^{n} p_i x_i\right)^m}{\left(\sum_{i=1}^{n} p_i\right)^{m-1}}. \tag{2.11}$$

Note 3. Putting $x_{i1}=\cdots=x_{im}=C$ ($C=$const) in inequality (2.5) we obtain

$$\sum_{i=1}^{n} \frac{p_i}{y_{i1}\cdots y_{i,m-1}} \geq \frac{\left(\sum_{i=1}^{n} p_i\right)^m}{\sum_{i=1}^{n} p_i y_{i1}\cdots \sum_{i=1}^{n} p_i y_{i,m-1}}; \tag{2.12}$$

especially, for $y_{i1}=\cdots=y_{i,m-1}=y_i$

$$\sum_{i=1}^{n} \frac{p_i}{(y_i)^{m-1}} \left(\sum_{i=1}^{n} p_i y_i\right)^{m-1} \geq \left(\sum_{i=1}^{n} p_i\right)^m \tag{2.13}$$

which is DUNKEL's inequality.

Combining (2.10), (2.11), (2.12) and (2.13) we obtain

$$\left(\sum_{i=1}^{n} p_i\right)^{m-1} \sum_{i=1}^{n} p_i x_{i1}\cdots x_{im} \geq \sum_{i=1}^{n} p_i x_{i1}\cdots \sum_{i=1}^{n} p_i x_{im} \geq \frac{\left(\sum_{i=1}^{n} p_i\right)^{m+1}}{\sum_{i=1}^{n} \frac{p_i}{x_{i1}\cdots x_{im}}} \tag{2.14}$$

and

$$\left(\sum_{i=1}^{n} p_i\right)^{m-1} \sum_{i=1}^{n} p_i x_i^m \geq \left(\sum_{i=1}^{n} p_i x_i\right)^m \geq \frac{\left(\sum_{i=1}^{n} p_i\right)^{m+1}}{\sum_{i=1}^{n} \frac{p_i}{x_i^m}}. \tag{2.15}$$

Note 4. If we put $x_{i1}=\cdots=x_{im}=x_i y_i$ and $y_{i1}=\cdots=y_{i,m-1}=y_i$ in inequality (2.5), where

$$\frac{x_{i1}\cdots x_{im}}{y_{i1}\cdots y_{i,m-1}}=y_i x_i^m$$

and

$$\frac{x_{i1}\cdots x_{i,j-1}}{y_{i1}\cdots y_{i,j-1}}=x_i^{m-1},\quad \frac{y_{i,j-1}}{x_{ij}}=\frac{1}{x_i},$$

we obtain

$$\left(\sum_{i=1}^{n} p_i x_i y_i\right)^m \leqq \sum_{i=1}^{n} p_i y_i x_i^m \left(\sum_{i=1}^{n} p_i y_i\right)^{m-1} \tag{2.16}$$

which is JENSEN's inequality.

If we substitute $x_{i1}=\cdots=x_{im}=x_i y_i$ and $y_{i1}=\cdots=y_{i,m-1}=y_i^{\frac{m}{m-1}}$ in inequality (2.5) we obtain

$$\sum_{i=1}^{n} p_i x_i^m \geqq \frac{\left(\sum_{i=1}^{n} p_i x_i y_i\right)^m}{\left(\sum_{i=1}^{n} p_i (y_i)^{\frac{m}{m-1}}\right)^{m-1}} \tag{2.17}$$

i.e.

$$\sum_{i=1}^{n} p_i x_i y_i \leqq \left(\sum_{i=1}^{n} p_i (x_i)^m\right)^{\frac{1}{m}} \left(\sum_{i=1}^{n} p_i (y_i)^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \tag{2.18}$$

which is HÖLDER's inequality.

For $m=2$ we obtain SCHWARTZ's inequality

$$\sum_{i=1}^{n} p_i x_i y_i \leqq \left(\sum_{i=1}^{n} p_i (x_i)^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i (y_i)^2\right)^{1/2}.$$

Note 5. Inequality (2.5) for $m=3$ and $x_{i3}=C$ ($C=$const) becomes

$$\sum_{i=1}^{n} p_i \frac{x_{i1} x_{i2}}{y_{i1} y_{i2}} \leqq \sum_{i=1}^{n} p_i \frac{\sum_{i=1}^{n} p_i x_{i1} \sum_{i=1}^{n} p_i x_{i2}}{\sum_{i=1}^{n} p_i y_{i1} \sum_{i=1}^{n} p_i y_{i2}} \tag{2.19}$$

where $\frac{x_{i1} x_{i2}}{y_{i1} y_{i2}}$ and y_{i2} are positive sequences of the same sense.

If we substitute in (2.19)

$$x_{i1}=x_{i1} y_{i1},\quad x_{i2}=x_{i2} y_{i2},$$

$$y_{i1}=x_{i1} y_{i2}\quad y_{i2}=x_{i2} y_{i1}$$

we obtain

$$\sum_{i=1}^{n} p_i \leqq \sum_{i=1}^{n} p_i \frac{\sum_{i=1}^{n} p_i x_{i1} y_{i1} \sum_{i=1}^{n} p_i x_{i2} y_{i2}}{\sum_{i=1}^{n} p_i x_{i1} y_{i2} \sum_{i=1}^{n} p_i x_{i2} y_{i1}},$$

i.e.

$$\sum_{i=1}^{n} p_i x_{i1} y_{i2} \sum_{i=1}^{n} p_i x_{i2} y_{i1} \leqq \sum_{i=1}^{n} p_i x_{i1} y_{i1} \sum_{i=1}^{n} p_i x_{i2} y_{i2} \tag{2.20}$$

under condition that $\frac{x_{i1}}{x_{i2}}$ and $\frac{y_{i1}}{y_{i2}}$ are of the same sense. This is FUJIWARA's inequality.

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NEKE NEJEDNAKOSTI ZA KONVEKSNE FUNKCIJE

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Koristeći osobine konveksnih funkcija u radu se daju generalizacije izvesnog broja u literaturi poznatih nejednakosti. Razmatrani su i diskretni slučajevi.

Na primer, dokazan je sledeći rezultat:

Teorema. *Ako su* $f_1, \ldots, f_n$ *i* $\Phi_1, \ldots, \Phi_{n-1}$ *integrabilne funkcije,* $f_1, \dfrac{f_k}{\Phi_{k-1}}$ $(k=2, \ldots, n)$ *konveksne funkcije,* Φ_k $(k=1, \ldots, n-1)$ *opadajuće na* $[a, b]$, $f_k(x)>0$ *za* $x \in (a, b]$, $f_k(a)=0$, $\Phi_k(x)>0$ *na* $[a, b]$ *i* p *pozitivna i integrabilna funkcija na* $[a, b]$, *tada važi nejednakost*

$$(1) \qquad \int_a^b p(x) \frac{f_1(x) \cdots f_n(x)}{\Phi_1(x) \cdots \Phi_{n-1}(x)} \mathrm{d}x \geqq M \frac{\int_a^b p(x) f_1(x) \mathrm{d}x \cdots \int_a^b p(x) f_n(x) \mathrm{d}x}{\int_a^b p(x) \Phi_1(x) \mathrm{d}x \cdots \int_a^b p(x) \Phi_{n-1}(x) \mathrm{d}x},$$

gde je

$$M = \frac{\int_a^b p(x)(x-a)^n \mathrm{d}x \left(\int_a^b p(x) \mathrm{d}x\right)^{n-1}}{\left(\int_a^b p(x)(x-a) \mathrm{d}x\right)^n}.$$

Jednakost u (1) *važi ako i samo ako je* $f_k(x) = C_k(x-a)$ $(k=1, \ldots, n)$ *i* $\Phi_k(x) = C$ $(k=1, \ldots, n-1)$.