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686. THE TRIANGLE INEQUALITY OF LESSELLS AND PELLING*

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In 1974, on the basis of a computer check of 500 random triangle, J. GARFUNKEL [1] conjectured the following interesting inequality. If α_a, β_b, μ_c, and s denote the altitude to side a, the bisector of angle B, the median to side c, and the semiperimeter $(a+b+c)/2$, then $\alpha_a+\beta_b+\mu_c \leqq s\sqrt{3}$. In 1976, C. S. GARDNER (among several others, see [1]) proved this by means of a sequence of elementary transformations and the differentiability of a function. He also pointed out that equality holds if and only if $a=b=c$. Recently, G. S. LESSELLS and M. J. PELLING [2] studied the following stronger inequality:

(1) $$\beta_a+\beta_b+\mu_c \leqq s\sqrt{3},$$

with equality if and only if $a=b=c$. The proof given depended upon transforming the functions

$$\beta_a=\frac{2\sqrt{bcs(s-a)}}{b+c}, \quad \mu_c=\left(\frac{a^2}{2}+\frac{b^2}{2}-\frac{c^2}{4}\right)^{1/2},$$

by setting $s=1$, $x=1-a$, $y=1-b$, to make

$$\beta(x, y)=\frac{2\sqrt{x(1-y)(x+y)}}{1+x}, \quad \mu(x, y)=\left(1-(x+y)+\frac{(x-y)^2}{4}\right)^{1/2}.$$

In these terms, which will be used henceforth, the inequality (1) becomes

$$\beta(x, y)+\beta(y, x)+\mu(x, y) \leqq \sqrt{3}$$

on $R=\{(x, y) \mid 0 \leqq x,\ y,\ x+y \leqq 1\}$, with equality only at $(x, y)=(1/3,\ 1/3)$, which corresponds to the case of the equilateral triangle. While LESSELLS and PELLING continued, with the use of some hours of computer time, to show the truth of their assertion, it is possible to demonstrate it by proving a stronger inequality.

Since the geometric mean is bounded by the arithmetic mean,

$$2\sqrt{bc}/(b+c) \leqq 1.$$

This means that (1) is weaker than

(2) $$\sqrt{s(s-a)}+\sqrt{s(s-b)}+\mu_c \leqq s\sqrt{3},$$

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with equality if and only if $a=b=c$, when as well $2\sqrt{bc}/(b+c)=1$. Under the transformation of LESSELLS and PELLING (2) becomes

$$g(x, y)=\sqrt{x}+\sqrt{y}+\mu(x, y)\leqq\sqrt{3},$$

with equality only for $g(1/3, 1/3)$. The function $g(x, y)$ has a domain properly containing R.

Theorem. *The following inequalities holds*

$$\alpha_a+\beta_b+\mu_c\leqq\beta_a+\beta_b+\mu_c\leqq\sqrt{s(s-a)}+\sqrt{s(s-b)}+\mu_c\leqq s\sqrt{3}, \tag{3}$$

with equality if and only if $a=b=c$.

Proof. After the above discussion, all that remains to be proved is that $g(x, y)$ has its unique maximum at $(1/3, 1/3)$. This follows in two steps. On the line segment $(0, 0)$ to $(1/2, 1/2)$, dividing R in half, $g(x, x)$ rises from 1 to $\sqrt{3}$ at $(1/3, 1/3)$ and then falls to $\sqrt{2}$, as can be checked by differentiation. The second step is to see that $g(x, y)$ falls away symmetrically from the values $g(x, x)$ in the remainder of R; $y=x$ specifies a ridge of the function. This can be seen as follows.

Every point of R, except the origin and $(1/2, 1/2)$, lies on a parabola $P(d)$, $\sqrt{x}+\sqrt{y}=\sqrt{d}$, for $0<d<2$. The other expression for $P(d)$,

$$(x-y)^2-2d(x+y)+d^2=0,$$

allows us to write $x+y=\frac{(x-y)^2+d^2}{2d}$.

Now, on $P(d)$, using this substitution for $x+y$ and $\sqrt{d}$ for $\sqrt{x}+\sqrt{y}$ in the expression for $g(x, y)$ shows

$$g(x, y)=\sqrt{d}+\frac{1}{2}[4-2d+(d-2)(x-y)^2/d]^{1/2}.$$

Everything in the final expression is constant except $(x-y)^2$, and its coefficient is negative since $d<2$.

On every parabola $P(d)$, the function $g(x, y)$ has its maximum at the vertex of $P(d)$, which lies on the line segment from $(0, 0)$ to $(1/2, 1/2)$. On the line segment, $g(x, x)$ has its maximum $\sqrt{3}$, at $(1/3, 1/3)$. The proof is complete.

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LESSELIS-PELLINGOVA NEJEDNAKOST ZA TR)UGAO

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Dokazana je nejednak6st (3) koja u sebi sadrži nejednakost LESSELLSa i PELLINGA.