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682. INEQUALITIES FOR DIVIDED DIFFERENCES*

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1. If $-\infty<a<b<+\infty$ and $f:[a, b]\to \mathbf{R}$ let us denote by $[x_0, x_1, \ldots, x_n; f]$ the divided difference at the system of knots

$$(d): \qquad a=x_0<x_1<\cdots<x_{n-1}<x_n=b.$$

For $f\in C^{(n)}[a, b]$, $g\in C[a, b]$, $g>0$ and $p\in(1, +\infty)$, we shall use the notation

$$\|f^{(n)}\|_p=\left(\frac{1}{b-a}\int_a^b |f^{(n)}(t)|^p\,\mathrm{d}t\right)^{1/p} \text{ and } \|f^{(n)}\|_{g,p}=\left(\int_a^b g(t)\,|f^{(n)}(t)|^p\,\mathrm{d}t\right)^{1/p}.$$

Our aim is to find the best upper bound $K_n^*=K_n^*(d, g, p)$ in the inequality

$$|[a, x_1, \ldots, x_{n-1}, b; f]|\leqq K_n^*\,\|f^{(n)}\|_{g,p}, \qquad n\geqq 2.$$

2. Let us put $w(x)=w(x:d)=\prod_{j=0}^{n}(x-x_j)$ and

$$(1) \qquad Q_n(t)=Q_n(t:d)=\frac{1}{(n-1)!}\sum_{i=0}^{n}\frac{(x_i-t)_+^{n-1}}{w'(x_i)}, \quad t\in[a, b],$$

where

$$(x-t)_+^{n-1}=\begin{cases}(x-t)^{n-1}, & a\leqq t\leqq x\leqq b\\ 0, & a\leqq x<t\leqq b.\end{cases}$$

We first investigate the case $g=1$ and $p=2$.

Theorem 1. *If* $f\in C^{(n)}[a, b]$ *and* $a<x_1<\cdots<x_{n-1}<b$, *then*

$$(2) \qquad |[a, x_1, \ldots, x_{n-1}, b; f]|\leqq K_n(d)\,\|f^{(n)}\|_2$$

where

$$(3) \qquad K_n(d)=\left(\frac{(-1)^n(b-a)}{(2n-1)!}\sum_{k=0}^{n-1}\sum_{i=k+1}^{n}\frac{(x_i-x_k)^{2n-1}}{w'(x_i)\cdot w'(x_k)}\right)^{1/2}.$$

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If

(4) $$f(t)=f_*(t)=C\sum_{i=0}^{n}\frac{(x_i-t)_+^{2n-1}}{w'(x_i)}+P_{n-1}(t),$$

P_{n-1} being an arbitrary polynomial of the degree $n-1$ and C a real constant, then in (2) *holds the equality.*

Proof. If $f\in C^{(n)}[a, b]$ then

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k+\int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt.$$

In other words

$$f(x)=P_{n-1}(x)+\int_a^b f^{(n)}(t)\frac{(x-t)_+^{n-1}}{(n-1)!}\,dt,$$

where P_{n-1} is a polynomial of the degree $n-1$.

Since $[a, x_1, \ldots, x_{n-1}, b; P_{n-1}]=0$ one obtains

(5) $$[a, x_1, \ldots, x_{n-1}, b; f]=\int_a^b f^{(n)}(t)Q_n(t)\,dt$$

where Q_n is defined as in (1) If in (4) $C=\frac{(-1)^n}{(2n-1)!}$, then $f_*^{(n)}=Q_n$. Therefore (5) enables us to write

$$\int_a^b Q_n^2(t)\,dt=[a, x_1, \ldots, x_{n-1}, b; f_*]$$

$$=\frac{(-1)^n}{(2n-1)!}\sum_{k=0}^{n-1}\sum_{i=0}^{n}\frac{(x_i-x_k)_+^{2n-1}}{w'(x_i)\cdot w'(x_k)}=\frac{1}{b-a}[K_n(d)]^2.$$

According to (5)

$$|[a, x_1, \ldots, x_{n-1}, b; f]|\le\left(\int_a^b Q_n^2(t)\,dt\right)^{1/2}\cdot\left(\int_a^b |f^{(n)}(t)|^2\,dt\right)^{1/2}$$

$$=K_n(d)\cdot\|f^{(n)}\|_2,$$

with equality if and only if $f^{(n)}(t)=C_1\cdot Q_n(t)$, where C_1 is a real constant.

Taking into account that $K_2(d)=1/\sqrt{3}$, the following proposition is true:

Theorem 2. *if* $f \in C^{(2)}[a, b]$ *and* $a<x<b$, *then*

$$|[a, x, b; f]| \leqq \frac{1}{\sqrt{3}} \|f''\|_2,$$

with equality if and only if f is given by

$$f(t)=\begin{cases} C\dfrac{(b-t)^3}{b-x}+C\dfrac{(t-x)^3(b-a)}{(b-x)(x-a)}+C_1 t+C_2, & t\in[a, x] \\ C\dfrac{(b-t)^3}{b-x}+C_1 t+C_2, & t\in(x, b] \end{cases}$$

where C, C_1, C_2 *are arbitrary real constants.*

This theorem is an extension of a result established by V. A. Zmorovič [4] in the case $x=(a+b)/2$ (see also [1] and [3], page 300). Let us assume that $f(a)=f(b)=0$. From the above theorem one finds the inequality

$$|f(x)| \leqq \frac{(x-a)(b-x)}{\sqrt{3}} \|f''\|_2 .$$

Taking into account that $(x-a)(b-x) \leqq \frac{(b-a)^2}{4}$, $x \in [a, b]$, we obtain $|f(x)| \leqq \frac{(b-a)^2}{4\sqrt{3}} \|f''\|_2$, $x\in[a, b]$. Therefore, with the notation $\|f\|_\infty = \max\limits_{x\in[a,b]} |f(x)|$ we have proved the following

Theorem 3. *If* $f\in C^{(2)}[a, b]$, $f(a)=f(b)=0$, *then*

$$\|f\|_\infty \leqq \frac{(b-a)^2}{4\sqrt{3}} \|f''\|_2,$$

the equality case being valid for

$$f(t)=\begin{cases} c(t-a)^3-\dfrac{3}{4}c(t-a)(b-a)^2, & t\in[a, (a+b)/2] \\ c(b-t)^3-\dfrac{3}{4}c(b-t)(b-a)^2, & t\in((a+b)/2, b], \end{cases}$$

where c is an arbitrary real constant.

Another particular case of the inequality (2) is the following: if the points $x_k \in (d)$ are equidistant, that is $x_k = a+\frac{k}{n}(b-a)$, $k=0, 1, \ldots, n$ then

$$[a, x_1, \ldots, x_{n-1}, b; f] = \frac{n^n}{n!\,(b-a)^n} \Delta^n f(a)$$

where $\Delta^n f(a) = \sum\limits_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f\left(a+\frac{k}{n}(b-a)\right)$. From theorem 1 one finds the following proposition

Theorem 4. *$f \in C^{(n)}[a, b]$, then*

$$|\Delta^n f(a)| \leq (b-a)^n \cdot C_n^* \| f^{(n)} \|_2$$

where

$$C_n^* = \frac{1}{n^{n-1}} \left(2 \sum_{k=1}^{n} \frac{(-1)^{n-k} k^{2n-1}}{(n+k)!\,(n-k)!} \right)^{\frac{1}{2}}.$$

Moreover

$$C_2^* = \frac{1}{\sqrt{12}},\quad C_3^* = \frac{1}{18}\sqrt{\frac{11}{15}},\quad C_4^* = \frac{1}{384}\sqrt{\frac{151}{35}},\quad C_5^* = \frac{1}{45\,000}\sqrt{\frac{15\,619}{35}}.$$

In order to find this form of the constant C_n^* we note that the following equality

$$\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} (-1)^{i+k} \binom{n}{i} \binom{n}{k} (i-k)^{2n-1} = \sum_{k=1}^{n} \binom{2n}{n-k} (-1)^k k^{2n-1}$$

is true.

3. Let $p>1$ and $Q_n(t) = Q_n(t:d)$ be defined as in (1). Taking into account that $Q_n \geq 0$ (see [2]) let us denote

$$K_n^* = K_n^*(d,\ g,\ p) = \left(\int_a^b \frac{[Q_n(t)]^{p/(p-1)}}{[g(t)]^{1/(p-1)}}\, dt \right)^{(p-1)/p} \tag{6}$$

where $g \in C[a, b]$ is a positive function. By means of HÖLDER's inequality, from (5) we obtain

$$|[a,\ x_1, \ldots, x_{n-1},\ b]| \leq \int_a^b |f^{(n)}(t)|\,[g(t)]^{1/p} \cdot \frac{Q_n(t)}{[g(t)]^{1/p}}\, dt$$

$$\leq \| f^{(n)} \|_{g,p} \left(\int_a^b \frac{[Q_n(t)]^q}{[g(t)]^{q/p}}\, dt \right)^{1/q},\quad \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore the folowing result is true:

Theorem 5. *If $f \in C^{(n)}[a, b]$ and $(d): a < x_1 < \cdots < x_{k-1} < b$ is an arbitrary sistem of knots, then*

$$|[a,\ x_1, \ldots, x_{n-1},\ b;\ f]| \leq K_n^* \| f^{(n)} \|_{g,p} \tag{7}$$

where K_n^* is defined in (6). If the function f verifies

$$f^{(n)}(t) = C \left(\frac{Q_n(t:d)}{g(t)} \right)^{1/(p-1)},$$

then in (7) holds the equality.

The case $n=2$, $x_1 = \frac{1}{2}(a+b)$ was investigated by R. Ž. ĐORĐEVIĆ and G. V. MILOVANOVIĆ [1].

Using the fact that for $g=1$

$$K_2^*(d,\ 1,\ p) = \frac{1}{(b-a)^{1/p}} \left(\frac{p-1}{2p-1} \right)^{\frac{p-1}{p}}$$

we obtain the following particular case:

Theorem 6. *If* $f \in C^{(2)}[a, b]$ *and* $a<x<b$, $p>1$, *then*

$$|[a, x, b; f]| \leqq \left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \cdot \|f''\|_p, \tag{8}$$

with equality for

$$f(t)=\begin{cases} C\left(\frac{(t-a)^{2p-1}}{x-a}\right)^{\frac{1}{p-1}}+C_1 t+C_2, & a \leqq t \leqq x \\ C\left(\frac{(b-t)^{2p-1}}{b-x}\right)^{\frac{1}{p-1}}+C_1 t+C_2+C(b-a)\left(2t-a-b+\frac{t-x}{p-1}\right), & x<t \leqq b. \end{cases}$$

If f is an element from $C^{(2)}[a, b]$ with $f(a)=f(b)=0$, then (8) furnishes us

$$|f(x)| \leqq (x-a)(b-x)\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \|f''\|_p \leqq \frac{(b-a)^2}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \|f''\|_p .$$

This delivers

Theorem 7. *If* $f \in C^{(2)}[a, b]$, $f(a)=f(b)=0$, *then*

$$\|f\|_\infty \leqq \frac{(b-a)^2}{4}\left(\frac{p-1}{2p-1}\right)^{\frac{p-1}{p}} \cdot \|f''\|_p$$

with equality for

$$f(t)=C\begin{cases} \left(\frac{2}{b-a}\right)^{\frac{1}{p-1}}(t-a)^{\frac{2p-1}{p-1}}-(t-a)(b-a)\frac{2p-1}{2(p-1)}, & a \leqq t \leqq \frac{a+b}{2} \\ \left(\frac{2}{b-a}\right)^{\frac{1}{p-1}}(b-t)^{\frac{2p-1}{p-1}}-(b-t)(b-a)\frac{2p-1}{2(p-1)}, & \frac{a+b}{2}<t \leqq b. \end{cases}$$

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NEJEDNAKOSTI ZA DEONE RAZLIKE

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Predmet ovog rada su najbolje moguće nejednakosti za deone razlike. Dobijeni rezultati dati su u obliku teorema 1—6.