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681. THE SYMMETRIC, LOGARITHMIC AND POWER MEANS*

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Let $L(x, y)$ denote the logarithmic mean of positive x and y:

$$L(x, y)=\frac{x-y}{\log(x)-\log(y)} \qquad (x\neq y). \tag{1}$$

In two recent notes [1] and [3], the relationship of $L(x, y)$ to the p-th arithmetic or power mean

$$M_p(x, y)=\left[\frac{1}{2}(x^p+y^p)\right]^{\frac{1}{p}}, \; p\neq 0, \tag{2}$$

was discussed and proofs of the following results established. Let $M_0(x, y)$ denote the geometric mean $\sqrt{x\cdot y}$. Then for positive $x\neq y$

$$M_0(x, y)<\frac{1}{2}(x^{3/4}y^{1/4}+x^{1/4}y^{3/4})<L(x, y)<M_p(x, y), \tag{3}$$

provided $\frac{1}{3}\leqq p$. The third inequality in (3) is sharp in the sense that if

$$0<p<\frac{1}{3}, \quad L(x, y)<M_p(x, y)$$

for some, but not all, positive x and y.

The sharpness of $p=1/3$ was shown in [3], and it would be interesting to obtain an analogous result for the lower bound. At this point cognoscenti of HARDY, LITTLEWOOD and POLYA [2] may recognize the second expression in (3) as an example of the symmetric mean of positive x and y:

$$S_\delta(x, y)=\frac{1}{2}\left(x^{\frac{1+\sqrt{\delta}}{2}}\cdot y^{\frac{1-\sqrt{\delta}}{2}}+x^{\frac{1-\sqrt{\delta}}{2}}\cdot y^{\frac{1+\sqrt{\delta}}{2}}\right), \tag{4}$$

where we have used the rather unnatural form of the exponent for reasons which will become clear below.

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It is shown in [2: II. 46, 47, 48] that S_δ is increasing in δ and that for $x \neq y$

$$M_0(x, y) < S_\delta(x, y) < M_1(x, y)$$

provided $0 < \delta < 1$. Since the second inequality in (3) involves $S_{1/4}$, one might expect a sharp upper bound on δ analogous to the lower bound on p. In this note we show that $\delta = 1/3$ is such an upper bound and do so by means of an elementary proof which can be easily modified to give $1/3$ as a sharp lower bound for p.

To simplify our statements, introduce the following

Definition. *F and G will be called comparable on a domain R if one of the inequalites $F(z) \leqq G(z)$ or $F(z) \geqq G(z)$ holds for all z in R.*

Theorem 1. *Let $x \neq y$ be positive and $0 \leqq \delta \leqq 1/3 \leqq p \leqq 1$. Then*

$$S_\delta(x, y) < L(x, y) < M_p(x, y). \tag{5}$$

If $\frac{1}{3} < \delta < 1$ $\left(0 < p < \frac{1}{3}\right)$, L is not comparable to S_δ (resp. M_p).

In proving the theorem we obtain an equivalent result which is recorded as a

Corollary. *Suppose $t > 0$. Then for $0 \leqq \delta \leqq 1/3 \leqq p \leqq 1$,*

$$t \cdot \cosh(\sqrt{\delta}\, t) < \sinh(t) < t(\cosh(pt))^{1/p}. \tag{6}$$

If $\frac{1}{3} < \delta < 1$ $\left(0 < p < \frac{1}{3}\right)$, $\sinh(t)$ *is not comparable to* $t \cosh(\sqrt{\delta}\, t)$ (resp. $t \cdot (\cosh(pt))^{1/p}$).

Proof. It is clear from the definitions of S_δ, L and M_p that if $y = 1$, (5) will be valid for sufficiently large values of x. We shall show that only for δ and p in the prescribed range will (5) be valid for all x and y.

For the first inequality assume $0 < y < x$ and divide through by y:

$$\frac{1}{2}\left[\left(\frac{x}{y}\right)^{\frac{1+\sqrt{\delta}}{2}} + \left(\frac{x}{y}\right)^{\frac{1-\sqrt{\delta}}{2}}\right] < \left(\frac{x}{y} - 1\right) \Big/ \log\left(\frac{x}{y}\right).$$

Using $e^{2t} = x/y$, multiply by te^{-t} to obtain the first inequality in (6):

$$t \cosh(\sqrt{\delta}\, t) < \sinh(t). \tag{7}$$

For (7) to be valid for small t it is necessary that

$$t\left(1 + \frac{\delta t^2}{2}\right) \leqq t + \frac{t^3}{6}$$

or that $\delta \leqq 1/3$. Note that if $1/3 < \delta$, then (7) is false for small t, and thus S_δ can't be comparable to L.

To prove $\delta \leqq 1/3$ is sufficient, consider the equivalent inequality

$$f_1(t) = \frac{\sinh(t)}{\cosh(\sqrt{\delta}\, t)} - t > 0.$$

Then $f_1'(t) = h_1(t)/2\cosh^2(\sqrt{\delta}\, t)$, where after the use of hyperbolic identities,

$$h_1(t) = (1-\sqrt{\delta})\cosh((1+\sqrt{\delta})\, t) + (1+\sqrt{\delta})\cosh((1-\sqrt{\delta})\, t) - \cosh(2\sqrt{\delta}\, t) - 1$$

$$= \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} A_k(\delta)$$

with $A_k(\delta) = (1-\delta)\,[(1+\sqrt{\delta})^{2k-1} + (1-\sqrt{\delta})^{2k-1}] - (4\delta)^k$.

Since S_δ increases with δ, it would suffice to show $A_k(1/3) \geqq 0$ and is strictly positive for some k. But $A_1(1/3) = 0$, and for $k > 1$

$$A_k\left(\frac{1}{3}\right) = \frac{4}{3} \sum_{j=1}^{k-1} \left(\frac{1}{3}\right)^j \left[\binom{2k-1}{2j} - \binom{k-1}{j}\right] > 0.$$

This verifies the first inequalities in (5) and (6).

For completeness we sketch a similar approach for the upper bounds. An identical substitution in the upper bound in (5) gives the upper bound in (6). Again examining small t gives

$$t + \frac{t^3}{6} \leqq t\left(1 + \frac{1}{p}\left(\frac{1}{2}(pt)^2\right)\right)$$

or $\frac{1}{3} \leqq p$. If $0 < p < \frac{1}{3}$, then L and M_p are not comparable.

Since M_p increases with p, we again need examine only the putative extreme value, and we begin with the equivalent inequality

$$f_2(t) = t - \sinh(t)/(\cosh(pt))^{1/p} > 0.$$

Differentiation gives $f_2'(t) = h_2(t)/8\,(\cosh(pt))^{1+1/p}$ with

$$h_2(t) = 8\,[(\cosh(pt))^{1+1/p} - \cosh((p-1)\, t)].$$

Using $p = \frac{1}{3}$ and some hyperbolic identities we find

$$h_2(t) = \cosh\left(\frac{4}{3}\, t\right) - 4\cosh\left(\frac{2}{3}\, t\right) + 3 = \sum_{k=1}^{+\infty} \left(\frac{2t}{3}\right)^{2k} \frac{1}{(2k)!}\,(4^k - 4).$$

The conclusion follows as before, thus completing the proof of the Theorem and of the Corollary.

One further question is raised, of course, and that is the comparability of S_δ and M_p.

Theorem 2. *Suppose* $0 \leqq \delta < 1$. *Then for positive* $x \neq y$

$$S_\delta(x, y) < M_p(x, y), \tag{8}$$

provided $\delta \leqq p$. *If* $0 < p < \delta < 1$, S_δ *and* M_p *are not comparable.*

If $\delta > 1$, *then*

$$S_\delta(x, y) > M_p(x, y), \tag{9}$$

provided $\delta \geqq p$. *If* $1 < \delta < p$, S_δ *and* M_p *are not comparable.*

Corollary. *Suppose* $0 \leqq \delta < 1$. *Then for* $t > 0$

$$\cosh(\sqrt{\delta}\, t) < (\cosh(pt))^{1/p}, \tag{10}$$

provided $\delta \leqq p$. *If.* $0 < p < \delta < 1$, *the two functions are not comparable.*

If $\delta > 1$, *then*

$$\cosh(\sqrt{\delta}\, t) > (\cosh(pt))^{1/p}, \tag{11}$$

provided $\delta \geqq p$. *If* $1 < \delta < p$, *the two functions are not comparable.*

Proof. It is clear again that the assorted inequalities are valid for $y = 1$ and large x. Our usual transformation gives (10) from (8) and forces for small t

$$1 + \frac{\delta t^2}{2} \leqq 1 + \frac{1}{p}\left(\frac{1}{2}(pt)^2\right) \tag{12}$$

or $\delta \leqq p$ as a necessary condition. Using the equivalent inequality

$$f_3(t) = 1 - \frac{\cosh(\sqrt{\delta}\, t)}{(\cosh(pt))^{1/p}} > 0, \tag{13}$$

we again obtain $f_3'(t)$ as $h_3(t)$ divided by a positive quantity, and

$$h_3(t) = (1 - \sqrt{\delta})\sinh(t(p + \sqrt{\delta})) - (1 + \sqrt{\delta})\sinh(t(\sqrt{\delta} - p)).$$

Using the extreme value $\delta = p < 1$ and $z = t\sqrt{p}$ gives for $h_3(z/\sqrt{p})$

$$\sum_{k=0}^{+\infty} \frac{z^{2k+1}}{(2k+1)!}[(1-\sqrt{p})(1+\sqrt{p})^{2k+1} - (1+\sqrt{p})(1-\sqrt{p})^{2k+1}]$$

$$= \sum_{k=1}^{+\infty} \frac{z^{2k+1}}{(2k+1)!} 2\sqrt{p}\,(1-p) \sum_{j=0}^{k-1} \binom{2k}{j} p^j,$$

completing the proof of (8) and (10).

Analogous arguments give (9) and (11) completing the proofs of the second theorem and corollary.

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SIMETRIČNA, LOGARITAMSKA I STEPENA SREDINA

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U ovom radu autor je dokazao nejednakosti (5) i (8) između simetričnih, logaritamskih i stepenih sredina.