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680. INEQUALITIES BETWEEN ARITHMETIC AND LOGARITHMIC MEANS*

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1. Introduction. In the analysis of the transmission of heat between two fluids, it is sometimes appropriate to use the logarithmic mean of the temperature differences, where that quantity is defined as $L=L(x, y)=(x-y)/(\log(x)-\log(y))$. (See, for example, [2].) Now if $A_p=A_p(x,y)=((x^p+y^p)/2)^{1/p}$ denotes the p-th arithmetic mean, then using $A_0=\sqrt{x\cdot y}$ it is easy to see that $A_0\leqq L\leqq A_1$. What is not as well known, and what was shown in [1], is that $L\leqq A_p$ if $p\geqq\frac{1}{3}$. Moreover 1/3 is sharp: if $0<p<1/3$ both $L(x, y)<A_p(x, y)$ and $L(x, y)>A_p(x, y)$ will occur.

It is also well known that A_p increases from the geometric mean A_0 to the arithmetic mean A_1 as p increases from 0 to 1. In [3] STOLARSKY defined a generalized logarithmic mean which is also an increasing interpolator from A_0 to A_1:

$$L_r=L_r(x, y)=[(x^r-y^r)/(r(x-y))]^{1/(r-1)}, \tag{1}$$

with the limiting values used at $x=y$ and at $r=0$ and $r=1$. Thus $A_0=L_{-1}$, $L=L_0$ and $A_1=L_2$. Since both L_r and A_p interpolate A_0 and A_1, it is natural to ask for inequalities between them. In this note we use elementary techniques to resolve that question completely. Since the proof is rather involved, most of the details are omitted.

2. Statement of the results. Let $a_1=a_1(r)=(r+1)/3$ and for $r>0$ let $a_2=a_2(r)=(r-1)\log(2)/\log(r)$ with $a_2(1)=\log(2)$. For $0<r$ define $p_1=\min(a_1, a_2)$ and $p_2=\max(a_1, a_2)$. For $-\infty<r<0$, $p_1=\min(0, a_1)$ and $p_2=\max(0, a_1)$.

Theorem. *For all positive x and y*

$$A_{p_1}(x, y)\leqq L_r(x, y)\leqq A_{p_2}(x, y), \tag{2}$$

p_1 and p_2 are sharp, and there is equality only when $x=y$ or else when r equals 2, 1/2, *or* -1.

Note that if $r=0$ we obtain $A_0\leqq L\leqq A_{1/3}$.

An equivalent result is

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Theorem′. *Suppose p_1 and p_2 are as above and $0 \leqq t < \infty$.*

Then

$$(3) \qquad (\cosh(p_1 t))^{1/p_1} \leqq [\sinh(rt)/(r \cdot \sinh(t))]^{1/(r-1)} \leqq (\cosh(p_2 t))^{1/p}$$

with limiting values at $r=1$ and $r=0$. Again p_1 and p_2 are sharp, and there is strict inequality for positive t unless r equals 2, 1/2 or -1.

Relations (2) and (3) are related as follows: assume $0<y\leqq x$ and set $e^{2t}=x/y$. Then division of (2) by $\sqrt{xy}$ gives (3). We remark that the monotonicity of $\log(L_r)$ can be verified easily using the middle term of (3).

3. Proof of Theorem′ for $0<r<\infty$. It is easy to check the cases $r=2$ and $r=1/2$ as well as the fact that the case $0<r\leqq 1$ follows from $1\leqq r<\infty$. With the limit expression understood for $r=1$, assume $1\leqq r$ and define

$$(4) \qquad f(t)=p^{-1}\cdot\log[\cosh(pt)]-(r-1)^{-1}\log[\sinh(rt)/(r\cdot\sinh(t))].$$

For small t, $f(t)\approx t^2(p-a_1)/2$, while for large t, $f(t)\approx(\log(r)/(r-1))(p-a_2)/p$, thus motivating the definitions of p_1 and p_2 as well as establishing the assertion about sharpness.

Next compute $f'(t)$ and use elementary identities to obtain $f'(t)=N(t)/D(t)$, where $D(t)$ is positive for positive t and

$$(5) \qquad N(t)=\sum_{k=1}^{\infty} c_k(r-1,\ p)\cdot t^{2k+1}/(2k+1)!$$

with

$$(6) \qquad c_k(x,\ u)=x^{-1}[(x+u)^{2k+1}+(x+1)(x-u)^{2k+1}-x(2+x-u)^{2k+1}].$$

The idea is to determine the sign of $f'(t)$ by examining the coefficients c_k. To this end note that for $0\leqq x<1$ $c_k(x,\ x)<0<c_k(x,\ 1)$, while for $1<x<\infty$, $c_k(x,\ 1)<0<c_k(x,\ x)$. Also $\partial c_k/\partial u>0$ if $0\leqq x,\ u\leqq 1$ or if $1<u\leqq x$.

Case 1: $0\leqq x=r-1<1$ and $p=(x+2)/3=a_1(r)$.

Then $c_1(x,\ p)=0$, and we will show $c_k(x,\ s)<0$, $k\geqq 2$, where $s=s(x)=(x+3)/4$. Since $x<p<s$, it then follows from $\partial c_k/\partial u>0$ that $c_k(x,\ p)<0$; hence $f'(t)<0$ and $f(t)<0$ for positive t.

Expanding in terms of x it is easy to check that

$$c_k(x,\ s)=(3/4)^{2k+1}\sum_{m=-k}^{k} x^{k+m}\cdot b_k(m)$$

where the sign of $b_k(m)$ depends on

$$\bar{b}_k(m)=(k+1-m)(5/3)^{k+1+m}-(k+1+m)(5/3)^{k+1-m}-2m(-1)^{k+m}.$$

Now $\overline{b}_k(0)=0, \overline{b}_k(-m)=-\overline{b}_k(m)$ and $\overline{b}_k(m)>0$ for $1\leqq m\leqq k$. Since $c_k(1,(s(1))=$ $=0$, it thus follows that $c_k(x, s)<0$ for $0\leqq x<1$.

Case 2: $0\leqq x=r-1<1$ and $p=x\cdot\log(2)/\log(x+1)$. Using the concavity of p, verify that $(x+2)/3<p<(x+3)/4$. By the results above $c_1(x, p)$ must now be positive and $c_k(x, p)$ negative for $k\geqq 2$. Hence $f'(t)$ is initially positive, has one zero and is ultimately negative. Since $f(0)=0=\lim_{t\to\infty} f(t)$ in this case, $f(t)>0$ for $0<t<\infty$.

Case 3: $1<x=r-1$ and $p=(x+2)/3$. Let $u_k(x)$ be defined uniquely by $c_k(x, u_k(x))=0$, so that $u_1(x)=p$. Since $u_k'(1)=(2k+1)^{-1}$, it follows that $u_{k+1}(x)<u_k(x)$ for x near 1. If that were true for all $x>1$, then $c_k(x, p)$ would be positive for $k\geqq 0$ and $f(t)$ positive for positive t.

Assume then that $c_k(x, u)=c_{k+1}(x, u)=0$ for $1<u<x$ and $k\geqq 1$. Then using $y=((u-1)/u)^{1/(2k+1)}$ it follows that

$$h(y)=(1+y)(1-y^{2k+1})^{2k+2}-(1-y)(1-y^{2k+2})^{2k+1}=0$$

for some $0<y<1$. But, after factoring out $(1-y)^{2k+2}\cdot(1+y+\cdots+y^{2k})^{2k+1}$, it is easy to see the resulting expression is positive for all $0<y<1$. Hence $u_k(x)>u_{k+1}(x)$ for $k\geqq 1$, completing this case.

Case 4: $1<x=r-1$ and $p=x\cdot\log(2)/\log(x+1)$. Now $u_1(x)>p$ so that $c_1(x,p)<0$, and for each x there is some $\overline{k}$ such that $c_k(x, p)\leqq 0$ for $1\leqq k\leqq\overline{k}$ and $c_k(x,p)>0$ for $k>\overline{k}$. Since f approaches zero as t approaches infinity, $\overline{k}<\infty$. Thus f' is initially negative, has one zero and is ultimately positive, thereby forcing $f(t)<0$ for positive t.

4. Proof of Theorem' for $r=0$. Proceed as before to obtain $p_1=0$ and $p_2=1/3$ as the best possible values for bounds. The verification of the lower bound is immediate. For the upper bound set $g(t)=t-\sinh(t)/(\cosh(t/3))^3$ and check that $g'(t)>0$ for $t>0$.

5. Proof of Theorem' for $-\infty<r<0$. The case $r=-1$ is immediate, and the assertion for $-\infty<r<-1$ follows from that for $-1<r<0$. Assume then that $-1<r<0$, define $f(t)$ as in (4), and check that a_1 is a candidate for the upper bound and no positive p will work for the lower bound. Again compute $'(t)$ to obtain (5) and (6). Since analysis of c_k doesn't seem to work, let $r=(-1-y)^{-1}$, so that $0<y<\infty$ and $p=y/3(1+y)$. Then substitution into (6) reduces the proof to checking that terms of the form

$$a_k(j)=\binom{2k+1}{j}(2^j-1)-3\binom{2k+1}{j-1}$$

are positive for $k\geqq 2$ and $1\leqq j\leqq 2k+1$. Again $f'(t)>0$ and the proof is complete.

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NEJEDNAKOSTI IZMEĐU ARITMETIČKE I LOGARITAMSKE SREDINE

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U ovom radu autor je dokazao nejednakosti (2) između aritmetičke i generalisane logaritamske sredine.