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679. SOME REMARKS ON THE STIRLING NUMBERS*

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1. The Stirling numbers $S_1(n, k)$, $S(n, k)$ of the first and second kind respectively, can be defined by

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n} S_1(n, k)\, x^k \tag{1.1}$$

and

$$x^n=\sum_{k=0}^{n} S(n, k)\, x(x-1)\cdots(x-k+1). \tag{1.2}$$

Also it is well known that $S_1(n, n-k)$ and $S(n, n-k)$ are polynomials in n of degree $2k$ and that, for $k \geqq 1$,

$$S_1(n, n-k)=S(n, n-k)=0 \qquad (0 \leqq n \leqq k). \tag{1.3}$$

It is proved in [4] that there exist two triangular arrays

$$(B_1(k, j)),\ (B(k, j)) \qquad (k=1, 2, \ldots;\ j=1, \ldots, k)$$

such that, for $k \geqq 1$,

$$S_1(n, n-k)=\sum_{j=1}^{k} B_1(k, j) \binom{n+j-1}{2k} \tag{1.4}$$

and

$$S(n, n-k)=\sum_{j=1}^{k} B(k, j) \binom{n+j-1}{2k}. \tag{1.5}$$

Moreover

$$B_1(k, j)=B(k, k-j+1) \qquad (1 \leqq j \leqq k). \tag{1.6}$$

By means of (1.4) and (1.5), $S_1(n, n-k)$ and $S(n, n-k)$ are defined as polynomials in n for arbitrary real or complex n.

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Substituting from (1.6) in (1.4) we get

$$S_1(n,\ n-k)=\sum_{j=1}^{k} B(k,\ k-j+1)\binom{n+j-1}{2k}=\sum_{j=1}^{k} B(k,j)\binom{n+k-j}{2k},$$

so that

$$S_1(-n+k,\ -n)=\sum_{j=1}^{k} B(k,j)\binom{-n+2k-j}{2k}=\sum_{j=1}^{k} B(k,j)\binom{n+j-1}{2k}.$$

Therefore, by (1.5)

$$S_1(-n+k,\ -n)=S(n,\ n-k) \tag{1.7}$$

and similarly

$$S(-n+k,\ -n)=S_1(n,\ n-k). \tag{1.8}$$

For references see [5].

2. We have also the representations

$$S_1(n,\ n-k)=\sum_{j=0}^{k-1} S_1'(k,\ j)\binom{n}{2k-j} \tag{2.1}$$

and

$$S(n,\ n-k)=\sum_{j=0}^{k-1} S'(k,\ j)\binom{n}{2k-j}. \tag{2.2}$$

The coefficients on the right are the numbers of Jordan [6, Ch. 4] and Ward [7]. For the notation used here see [2].

In (2.1) replace n by $-n+k$. Thus

$$S_1(-n+k,\ n)=\sum_{j=0}^{k-1} S_1'(k,\ j)\binom{-n+k}{2k-j}=\sum_{j=0}^{k-1}(-1)^j S_1(k,\ j)\binom{n+k-j-1}{2k-j}$$

$$=\sum_{j=0}^{k-1}(-1)^j S_1'(k,j)\sum_{t=j}^{k-1}\binom{n}{2k-t}\binom{k-j-1}{t-j}=\sum_{t=0}^{j-1}\binom{n}{2k-t}\sum_{j=0}^{t}(-1)^j S_1'(k,j)\binom{k-j-1}{t-j}.$$

Hence, by (1.7) and (2.2),

$$S'(k,\ t)=\sum_{j=0}^{t}(-1)^j S_1'(k,\ j)\binom{k-j-1}{t-j}. \tag{2.3}$$

Similarly we have

$$S_1'(k,\ t)=\sum_{j=0}^{t}(-1)^j S'(k,\ j)\binom{k-j-1}{t-j}. \tag{2.4}$$

For a different proof of (2.3) and (2.4) as well as other relations of this kind involving the other coefficients see [2].

3. The results of §1 suggest the following.

Let

$$\{f_k(x)\},\ \{f_{1,k}(x)\} \qquad (k=0,\ 1,\ 2,\ldots) \tag{3.1}$$

be two sequences of polynomials that satisfy

$$\deg f_k(x) = \deg f_{1,k}(x) = 2k \qquad (k=0,\ 1,\ 2,\ldots) \tag{3.2}$$

and

$$f_k(j) = f_{1,k}(j) = 0 \qquad (0 \leqq j \leqq k). \tag{3.2$'$}$$

We may put [3], for $k \geqq 1$,

$$f_k(x) = \sum_{j=1}^{k} b(k,\ j)\binom{x+j-1}{2k}, \quad f_{1,k}(x) = \sum_{j=1}^{k} b_1(k,\ j)\binom{x+j-1}{2k}. \tag{3.3}$$

We shall say that the sequences $\{f_k(x)\}$, $\{f_{1,k}(x)\}$ are *conjugate* provided

$$b_1(k,\ j) = b(k,\ k-j+1) \qquad (1 \leqq j \leqq k). \tag{3.4}$$

Substituting from (3.4) in the second of (3.3) we get

$$f_{1,k}(x) = \sum_{j=1}^{k} b(k,\ k-j+1)\binom{x+j-1}{2k} = \sum_{j=1}^{k} b(k,j)\binom{x+k-j}{2k}$$

$$= \sum_{j=1}^{k} b(k,\ j)\binom{-x+k+j-1}{2k}.$$

Thus $f_{1,k}(-x+k) = \sum_{j=1}^{k} b(k,\ j)\binom{x+j-1}{2k}$, so that

$$f_{1,k}(-x+k) = f_k(x) \qquad (k=1,\ 2,\ldots). \tag{3.5}$$

Conversely if we assume (3.5) then the above steps can be reversed to get (3.4).

This proves the following

Theorem. *Two sequences of polynomials* $\{f_k(x)\}$, $\{f_{1,k}(x)\}$ *that satisfy* (3.1), (3.2) *and* (3.2)$'$ *are conjugate if and only if* (3.5) *holds.*

Corollary. *The sequence* $\{f_k(x)\}$ *is self-conjugate if and only if*

$$f_k(-x+k) = f_k(x) \qquad (k=1,\ 2,\ldots) \tag{3.6}$$

or equivalently

$$b(k,\ k-j+1) = b(k,\ j) \qquad (1 \leqq j \leqq k). \tag{3.7}$$

4. As an example illustrating the corollary we take

$$f_k(x)=\binom{x}{k}\binom{x-1}{k}. \tag{4.1}$$

It is easily verified that $f_k(x)$ satisfies (3.6). Put

$$\binom{x}{k}\binom{x-1}{k}=\sum_{j=1}^{k} b(k, j)\binom{x+j-1}{2k}. \tag{4.2}$$

Multiply both sides of (4.2) by $(x-k)(x-k-1)/(k+1)^2$. Using the identity

$$(x-k)(x-k-1)=(x+j-2k-1)(x+j-2k-2)$$
$$+2(k-j+1)(x+j-2k-1)+(k-j)(k-j+1),$$

we find after some manipulation that $b(k, j)$ satisfies the recurrence

$$b(k+1, j)=(k-j+2)(k-j+3)\,b(k, j-2) \tag{4.3}$$
$$+2(k+j)(k-j+2)\,b(k, j-1)+(k+j)(k+j+1)\,b(k, j).$$

Also, by (3.6), we have

$$b(k, k-j+1)=b(k, j) \qquad (1 \leqq j \leqq k). \tag{4.4}$$

An explicit formula for $b(k, j)$ is obtained as a special case of the following general result [3, §7].

Let $f(x)$ be an arbitrary polynomial of degree k and put

$$f(x+y-1)=\sum_{j=0}^{k}\binom{x+j-1}{k} C_{k,j}(y). \tag{4.5}$$

Then

$$C_{k,j}(y)=\sum_{t=0}^{j}(-1)^{k-t}\binom{k+1}{t} f(y-j+t-1) \tag{4.6}$$

and conversely.

In (4.5) replace k by $2k$ and take $y=1$. Thus (4.5) becomes

$f(x)=\sum_{j=1}^{k}\binom{x+j-1}{2k} C_{2k,j}(1)$ an$L_r(x_{;\,k,j}(1)=\sum_{t=0}^{j}(-1)^{t}\binom{2k+1}{t} f(-j+t).$

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Finally, taking $f(x)=\binom{x}{k}\binom{x-1}{k}$, we get

$$b(k, j)=\sum_{t=0}^{j}(-1)^{t}\binom{2k+1}{t}\binom{-j+t}{k}\binom{-j+t-1}{k} \tag{4.7}$$

or, if we prefer,

$$b(k, j)=\sum_{t=1}^{j}(-1)^{j-t}\binom{2k+1}{j-t}\binom{k+t}{k}\binom{k+t-1}{k} \quad (k\geqq 1). \tag{4.8}$$

The sum on the right is Saalschützian [**1**, p. 9] and we find that

$$b(k, j)=\frac{k+1}{k}\binom{k}{j}\binom{k}{j-1}=\binom{k+1}{j}\binom{k-1}{j-1}. \tag{4.9}$$

Therefore finally we have

$$\binom{x}{k}\binom{x-1}{k}=\sum_{j=1}^{k}\frac{k+1}{k}\binom{x+j-1}{2k}\binom{k}{j}\binom{k}{j-1}=\sum_{j=0}^{k-1}\binom{x+j}{2k}\binom{k+1}{j+1}\binom{k-1}{j}. \tag{4.10}$$

The identity (4.10) can be verified by Saalschütz's theorem.

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NEKE PRIMEDBE O STIRLINGOVIM BROJEVIMA

L. C

U ovom radu uopštene su neke formule za Stirlingove brojeve prve i druge vrste.