UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 228—233.

673. SOME FUNCTIONAL EQUATIONS WITH SEVERAL UNKNOWN FUNCTIONS

Radovan R. Janić and *Josip E. Pečarić*

In this paper we shall give some generalizations of some results obtained in papers [**1**] and [**2**]. We shall also provide some analogous results.

1. In this part we shall use the following notations

$$\prod_{i=1}^{r}{}' x_i=\prod_{\substack{i=1\\ i\neq k}}^{r} x_i,\quad \sum_{i=1}^{r}{}' x_i=\sum_{\substack{i=1\\ i\neq k}}^{r} x_i,\quad \exp_a u=a^u.$$

First, we will consider the following functional equation

$$f(x_1\cdots x_r)=\sum_{k=1}^{r} x_k^{\alpha}\, f_k\Big(\prod_{i=1}^{r}{}' x_i\Big)+d x_1^{\alpha}\cdots x_r^{\alpha}\qquad (d,\ \alpha\in \mathbf{R}). \tag{1}$$

The continuous solution of this equation is given by

$$\begin{aligned} f(x)&=x^{\alpha}\Big((r-1)\,C\log x+\sum_{i=1}^{r} C_i-\frac{d}{r-1}\Big),\\ f_k(x)&=x^{\alpha}\Big(C\log x+C_k-\frac{d}{r-1}\Big)\qquad (k=1,\ldots,r), \end{aligned} \tag{2}$$

where C, C_k $(k=1,\ldots,r)$ are real constants.

Indeed, by substitutions of

$$F(x)=f(x^{1/(r-1)})/x^{\alpha/(r-1)},\quad F_k(x)=f_k(x)/x^{\alpha},\quad X_k=\prod_{i=1}^{r}{}' x_i\qquad (k=1,\ldots,r),$$

from (1) we obtain

$$F(X_1\cdots X_r)=F_1(X_1)+\cdots+F_r(X_r)+d. \tag{3}$$

The continuous solution of (3) is (see [**1**], equation (4.1))

$$F(x)=C\log x+\sum_{i=1}^{r} C_i-\frac{d}{r-1},\quad F_k(x)=C\log x+C_k-\frac{d}{r-1}\qquad (k=1,\ldots,r), \tag{4}$$

wherefrom we obtain (2).

Functional equation

$$f(x_1+\cdots+x_r)=\sum_{k=1}^{r}\exp_a x_k f_k\Big(\sum_{i=1}^{r}{}' x_i\Big)+d\exp_a\sum_{i=1}^{r} x_i \tag{5}$$

has a solution

$$f(x)=a^x\left(C(r-1)x+\sum_{k=1}^{r}C_k-\frac{d}{r-1}\right),$$

(6)

$$f_k(x)=a^x\left(Cx+C_k-\frac{d}{r-1}\right)\qquad (k=1,\dots,r),$$

where $C, C_1, \dots, C_r$ are real constants. This result is a generalization of equation (1.12.2°) from [3].

Functional equation

(7) $$f\left(\prod_{i=1}^{r}x_i\right)=\prod_{k=1}^{r}\left(f_k\left(\prod_{i=1}^{r}{}'\,x_i\right)^{x_k^\alpha}\right)\exp_d\left(\prod_{i=1}^{r}x_i^\alpha\right)$$

has a solution

$$f(x)=\exp_d\left(-\frac{x^\alpha}{r-1}\right)(C_1\cdots C_r x^{r-1})^{Cx^\alpha},$$

(8)

$$f_k(x)=\exp_d\left(-\frac{x^\alpha}{r-1}\right)(C_k x)^{Cx^\alpha}\qquad (k=1,\dots,r),$$

where $C, C_1>0, \dots, C_r>0$ are real constants.

Functional equation

(9) $$f(x_1+\dots+x_r)=\prod_{k=1}^{r}\left(f_k\left(\sum_{i=1}^{r}{}'\,x_i\right)^{\exp_a x_k}\right)\exp_d\left(\exp_a\sum_{i=1}^{r}x_i\right)$$

has a solution given by

$$f(x)=\exp_d\left(-\frac{a^x}{r-1}\right)(C_1\cdots C_r a^{(r-1)x})^{Ca^x},$$

(10)

$$f_k(x)=\exp_d\left(-\frac{a^x}{r-1}\right)(C_k a^x)^{Ca^x}\qquad (k=1,\dots,r),$$

where $C, C_1>0, \dots, C_r>0$ are real constants.

Functional equation

(11) $$f\left(\prod_{i=1}^{r}x_i\right)=\prod_{k=1}^{r}\left(f_k(x_k)^{\prod_{i=1}^{r}{}'\,x_i^\alpha}\right)\exp_d\left(\prod_{i=1}^{r}x_i^\alpha\right)$$

has a solution

$$f(x)=\exp_d\left(-\frac{x^\alpha}{r-1}\right)(C_1\cdots C_r x)^{Cx^\alpha},$$

(12)

$$f_k(x)=\exp_d\left(-\frac{x^\alpha}{r-1}\right)(C_k x)^{Cx^\alpha}\qquad (k=1,\dots,r),$$

where $C, C_1>0, \dots, C_r>0$ are arbitrary constants.

Functional equation

(13) $$f\left(\prod_{i=1}^{r}x_i\right)=\sum_{k=1}^{r}\left(f_k(x_k)\prod_{i=1}^{r}{}'\,x_i^\alpha\right)+d\prod_{i=1}^{r}x_i$$

has a solution

$$\text{(14)}\qquad \begin{aligned} f(x) &= x^\alpha\left(C\log(C_1\cdots C_r x)-\frac{d}{r-1}\right),\\ f_k(x) &= x^\alpha\left(C\log(C_k x)-\frac{d}{r-1}\right)\qquad (k=1,\dots,r),\end{aligned}$$

where C, $C_1>0,\dots,C_r>0$ are arbitrary constants.

Functional equation

$$\text{(15)}\qquad f\left(\sum_{i=1}^r x_i\right)=\prod_{k=1}^r\left(f_k(x_k)^{\exp_a\left(\sum_{i=1}^{r}{}' x_i\right)}\right)\exp_d\left(\exp_a\sum_{i=1}^r x_i\right)$$

has a solution

$$\text{(16)}\qquad \begin{aligned} f(x) &= \exp_d\left(-\frac{a^x}{r-1}\right)(C_1\cdots C_r a^x)^{Ca^x},\\ f_k(x) &= \exp_d\left(-\frac{a^x}{r-1}\right)(C_k a^x)^{Ca^x}\qquad (k=1,\dots,r),\end{aligned}$$

where C, $C_1>0,\dots,C_r>0$ are arbitrary constants.

Functional equation

$$\text{(17)}\qquad f\left(\sum_{i=1}^r x_i\right)=\sum_{k=1}^r f_k(x_k)\exp_a\left(\sum_{i=1}^{r}{}' x_i\right)+d\exp_a\sum_{i=1}^r x_i$$

has a solution given by

$$\text{(18)}\quad f(x)=a^x\left(Cx+\sum_{i=1}^r C_i-\frac{d}{r-1}\right),\ f_k(x)=a^x\left(Cx+C_k-\frac{d}{r-1}\right)\quad (k=1,\dots,r)$$

where C, $C_1,\dots,C_r$ are arbitrary real constants.

2. Now, we shall quote some extension of results from **[1]**, i. e., we shall consider the following functional equations:

$$\text{(19)}\qquad f(xy)=x^\alpha h(y)+y^\beta g(x)+dx^\alpha y^\beta,$$

$$\text{(20)}\qquad f(xy)=x^\alpha g(y)+y^\beta f(x)+dx^\alpha y^\beta,$$

$$\text{(21)}\qquad f(xy)=x^\alpha f(y)+y^\beta f(x)+dx^\alpha y^\beta,$$

where α, β, $d\in\mathbf{R}$. In **[1]**, the solutions of (19) and (20) for $d=0$ and $\alpha\neq\beta$, and (21) for $\alpha\neq\beta$ are given.

The solution of functional equation (19) is:

$$\text{(22)}\qquad \begin{aligned} f(x) &= \begin{cases} A(x^\beta-x^\alpha)+Bx^\beta, & \text{for } \alpha\neq\beta,\\ x^\alpha(A\log x+B), & \text{for } \alpha=\beta;\end{cases}\\ g(x) &= \begin{cases} A(x^\beta-x^\alpha)+Bx^\beta-Cx^\alpha-dx^\alpha, & \text{for } \alpha\neq\beta,\\ x^\alpha(A\log x+B-C-d), & \text{for } \alpha=\beta,\end{cases}\\ h(x) &= \begin{cases} A(x^\beta-x^\alpha)+Cx^\beta, & \text{for } \alpha\neq\beta,\\ x^\alpha(A\log x+C), & \text{for } \alpha=\beta;\end{cases}\end{aligned}$$

where A, B and C are real constants.

Indeed, the solution of functional equation

$$f(xy) = x^{\alpha} u(y) + y^{\beta} g(x) \tag{23}$$

for $\alpha \neq \beta$ is given in [1]. Using the substitution

$$u(y) = h(y) + dy^{\beta}$$

from solution of (23) we obtain (22) for $\alpha \neq \beta$. Using (2) (or (14)) for $r=2$, we can easily get (22) for $\alpha = \beta$.

The solution of equation (20) is

$$\begin{aligned} f(x) &= \begin{cases} A(x^{\beta} - x^{\alpha}) + Bx^{\beta}, & \text{for } \alpha \neq \beta, \\ x^{\alpha}(A \log x + B), & \text{for } \alpha = \beta; \end{cases} \\ g(x) &= \begin{cases} A(x^{\beta} - x^{\alpha}) - dx^{\beta}, & \text{for } \alpha \neq \beta, \\ x^{\alpha}(A \log x - d), & \text{for } \alpha = \beta; \end{cases} \end{aligned} \tag{24}$$

where A and B are real constants.

The solution of equation (21) is

$$f(x) = \begin{cases} A(x^{\beta} - x^{\alpha}) - dx^{\beta}, & \text{for } \alpha \neq \beta, \\ x^{\alpha}(A \log x - d), & \text{for } \alpha = \beta; \end{cases} \tag{25}$$

where A is a real constant.

Using the previous results, we can get the solution of the following functional equation:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(x_i y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i^{\alpha_i} h_{ij}(y_j) + y_j^{\beta_j} g_{ij}(x_i) + d_{ij} x_i^{\alpha_i} y_j^{\beta_j}) \tag{26}$$

where α_i, β_j, $d_{ij} \in \mathbf{R}$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, i. e. we have

$$\begin{aligned} f_{ij}(x) &= \begin{cases} A_{ij}(x^{\beta_j} - x^{\alpha_i}) + B_{ij} x^{\beta_j}, & \text{for } \alpha_i \neq \beta_j, \\ x^{\alpha_i}(A_{ij} \log x + B_{ij}), & \text{for } \alpha_i = \beta_j; \end{cases} \\ g_{ij}(x) &= \begin{cases} A_{ij}(x^{\beta_j} - x^{\alpha_i}) + B_{ij} x^{\beta_j} - (C_{ij} + d_{ij}) x^{\alpha_i}, & \text{for } \alpha_i \neq \beta_j, \\ x^{\alpha_i}(A_{ij} \log x + B_{ij} - d_{ij} - C_{ij}), & \text{for } \alpha_i = \beta_j, \end{cases} \\ h_{ij}(x) &= \begin{cases} A_{ij}(x^{\beta_j} - x^{\alpha_i}) + C_{ij} x^{\beta_j}, & \text{for } \alpha_i \neq \beta_j, \\ x^{\alpha_i}(A_{ij} \log x + C_{ij}), & \text{for } \alpha_i = \beta_j; \end{cases} \end{aligned} \tag{27}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$, where A_{ij}, B_{ij} and C_{ij} are real constants.

In the special cases, functional equations

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i^{\alpha} h(y_j) + y_j^{\beta} g(x_i) + dx_i^{\alpha} y_j^{\beta}), \tag{28}$$

(29) $$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_i^{\alpha} g(y_j)+y_j^{\beta} f(x_i)+d x_i^{\alpha} y_j^{\beta}\right),$$

(30) $$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_i^{\alpha} f(y_j)+y_j^{\beta} f(x_i)+d x_i^{\alpha} y_j^{\beta}\right),$$

have solutions (22), (24) and (25) respectively.

Now, we shall give some results, which are analogous to functional equation (19).

Functional equation

(31) $$f(x+y)=a^x h(y)+b^y g(x)+d a^x b^y$$

has a solution

(32) $$f(x)=\begin{cases} A(b^x-a^x)+Bb^x, & \text{for } a\neq b,\\ a^x(Ax+B), & \text{for } a=b;\end{cases}$$
$$g(x)=\begin{cases} A(a^x-b^x)+Bb^x-Ca^x-da^x, & \text{for } a\neq b,\\ a^x(Ax+B-C-d), & \text{for } a=b;\end{cases}$$
$$h(x)=\begin{cases} A(b^x-a^x)+Cb^x, & \text{for } a\neq b,\\ a^x(Ax+C), & \text{for } a=b;\end{cases}$$

where A, B and C are real constants.

Indeed, using the substitutions:

$$x=\log u,\ y=\log v,\ f(\log x)=F(x),\ h(\log x)=H(x) \text{ and } g(\log x)=G(x),$$

(31) becomes

$$F(uv)=a^{\log u} H(v)+b^{\log v} G(u)+da^{\log u} b^{\log v},$$

i.e.

$$F(uv)=u^{\log a} H(v)+v^{\log b} G(u)+du^{\log a} v^{\log b}.$$

Using the solution of equation (21), we obtain (32).

Analogously, we can get the following results:

Functional equation

(33) $$f(xy)=h(y)^{x^{\alpha}} g(x)^{y^{\beta}} d^{x^{\alpha} y^{\beta}}$$

has a solution

(34) $$f(x)=\begin{cases} A^{(x^{\beta}-x^{\alpha})} B^{x^{\beta}}, & \text{for } \alpha\neq\beta,\\ (Bx^D)^{x^{\alpha}}, & \text{for } \alpha=\beta;\end{cases}$$
$$g(x)=\begin{cases} A^{(x^{\beta}-x^{\alpha})} B^{x^{\beta}} (Cd)^{-x^{\alpha}}, & \text{for } \alpha\neq\beta,\\ (BC^{-1} d^{-1} x^D)^{x^{\alpha}}, & \text{for } \alpha=\beta;\end{cases}$$
$$h(x)=\begin{cases} A^{(x^{\beta}-x^{\alpha})} C^{x^{\beta}}, & \text{for } \alpha\neq\beta,\\ (Cx^D)^{x^{\alpha}}, & \text{for } \alpha=\beta;\end{cases}$$

where $A>0$, $B>0$, $C>0$ and D are arbitrary constants.

Functional equation

(35) $$f(x+y)=h(y)^{a^x}\,g(x)^{b^y}\,d^{a^x b^y}$$

has a solution

$$f(x)=\begin{cases} A^{(b^x-a^x)}\,B^{b^x}, & \text{for } a\neq b,\\ (BA^x)^{a^x}, & \text{for } a=b;\end{cases}$$

(36) $$g(x)=\begin{cases} A^{(b^x-a^x)}\,B^{b^x}\,(Cd)^{-a^x}, & \text{for } a\neq b,\\ (BC^{-1}\,d^{-1}\,A^x)^{a^x}, & \text{for } a=b;\end{cases}$$

$$h(x)=\begin{cases} A^{(b^x-a^x)}\,C^{b^x}, & \text{for } a\neq b,\\ (CA^x)^{a^x}, & \text{for } a=b;\end{cases}$$

where $A>0$, $B>0$ and $C>0$ are real constants.

Similarly, we can formulate the results analogous to functional equations (20), (21), (26), (28), (29) and (30).

REFERENCES

1. B. Martić et R. R. Janić: *Sur quelques généralisations des équations fonctionnelles* These Publications № **602** — № **633** (1978), 223—228.
2. Pl. Kannappan: *An application of differential equation in information theory.* Glasnik Matematički **14** (34) (1979), 269—274.

I. I. Stamate: *Équations fonctionnelles contenant plusieurs fonctions inconnues.* These Publications № **354** — № **356** (1971), 123—156.