UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634—№ 677 (1979), 214—220.

671. A TRANSFORMATION WHICH MAPS DERIVATIVES INTO DIFFERENCES

Bratislav Danković and *Miomir Stanković*

1. Introduction

Let $\mathbf{R}$ be the set of all real numbers, $\mathbf{N}$ the set of all natural numbers $\mathbf{N}_0=\{0\}\cup\mathbf{N}$. Further, let $C^\infty(\mathbf{R})$ be the set of real functions having continuous derivetives of all orders.

For real sequences $a=(a_0,\ a_1,\ldots)$ and $b=(b_0,\ b_1,\ldots)$ define the operations $+$, $\cdot$, and $\Delta^k(\cdot)$ in the usual way i.e.

$$a+b=(a_0+b_0,\ a_1+b_1,\ldots),$$

$$\lambda a=(\lambda a_0,\ \lambda a_1,\ldots),$$

$$\Delta^k(a_n)=(\Delta^k a_n)$$

where

$$\Delta a_m=a_{m+1}-a_m,\ \Delta^k=\Delta(\Delta^{k-1})\qquad (k=1,\ 2,\ldots).$$

In [1] we can find the following definition: The function

$$x^{(\lambda)}=\frac{\Gamma(x+1)}{\Gamma(x+1-\lambda)}$$

will be called factorial of the order λ, for all values of x and $\lambda\,(x,\ \lambda\in\mathbf{R})$ for which the gamma function exists. This definition comprises all cases if we agree that when the gamma function is infinite (for arguments $0,\ -1,\ -2,\ldots$),

$$x^{(\lambda)}=\lim_{\varepsilon\to 0}(x+\varepsilon)^{(\lambda)}.$$

From this definition immediatelly follow the following properties of the function $x^{(\lambda)}$:

$$\begin{aligned}
&\text{(i)} && m^{(n)}=m(m-1)\cdots(m-n+1)\ \text{ for } m>n, &&\\
&\text{(ii)} && m^{(n)}=0\ \text{ for } m<n, && (m,\ n\in\mathbf{N}_0)\\
&\text{(iii)} && m^{(m)}=m!. &&
\end{aligned}$$

In this paper we will define a transformation which maps the space $C^\infty(\mathbf{R})$ into the set of all real sequences and we will examine some of its properties and the end of this paper we will apply this transformation to solve some difference equations.

2. Basic definition and properties

Definition. *A-transformation of a function $f(u)\in C^\infty(\mathbf{R})$ is the sequence $Af(u)=(a_m)$ where a_m is defined by*

$$a_m=\frac{\mathrm{d}^m}{\mathrm{d}u^m}e^u f(u)\Big|_{u=0} \quad (m\in\mathbf{N}_0). \tag{1}$$

Theorem 1. *The following formula is valid*

$$A\left(C_1 f_1(u)+C_2 f_2(u)\right)=C_1 Af_1(u)+C_2 Af_2(u) \tag{2}$$

where C_1 and C_2 are arbitrary constants.

Proof. By definition we have

$$A\left(C_1 f_1(u)+C_2 f_2(u)\right)=\left(\frac{\mathrm{d}^m}{\mathrm{d}u^m}\left(e^u\left(C_1 f_1(u)+C_2 f_2(u)\right)\right)\Big|_{u=0}\right)$$

so that

$$A\left(C_1 f_1(u)+C_2 f_2(u)\right)=\left(C_1\frac{\mathrm{d}^m}{\mathrm{d}u^m}e^u f_1(u)\Big|_{u=0}\right)+\left(C_2\frac{\mathrm{d}^m}{\mathrm{d}u^m}e^u f_2(u)\Big|_{u=0}\right)$$
$$=C_1 Af_1(u)+C_2 Af_2(u)$$

which proves the theorem.

Theorem 2. *If we have $f(u, a)\in C^\infty(\mathbf{R})$ for fixed a and if $f(u)$ has continuous derivatives with respect to a for all $u\in\mathbf{R}$ then we have*

$$A\frac{\partial f(u, a)}{\partial a}=\frac{\partial}{\partial a}Af(u, a). \tag{3}$$

Proof. Since the following relation is valid

$$\frac{\partial^m}{\partial u^m}\left(e^u\frac{\partial}{\partial a}f(u, a)\right)\Big|_{u=0}=\frac{\partial^m}{\partial u^m}\left(\frac{\partial}{\partial a}e^u f(u, a)\right)\Big|_{u=0}$$
$$=\frac{\partial}{\partial a}\frac{\partial^m}{\partial u^m}e^u f(u, a)\Big|_{u=0}$$

in accordance with the definition we can conclude that (3) holds.

Theorem 3. *If $Af(u)=(a_m)$, then we have the following relations*

1° $Af^{(n)}(u)=\Delta^n Af(u)$

2° $A\int_0^u f(t)\,\mathrm{d}t=\left(\sum_{i=0}^{m-1}a_i\right)$

3° $Au^k f^{(n)}(u)=(m^{(k)}\Delta^n a_{m-k})$

$(k\in\mathbf{N}_0)$

4° $Au^k=(m^{(k)})$.

Proof. 1° Let us suppose that $A\dfrac{d^k f(u)}{du^k}=(b_m^k)$ and $k=1$. Then the following formula

$$a_{m+1}=\frac{d^{m+1}}{du^{m+1}}e^u f(u)\Big|_{u=0}=\frac{d^m}{du^m}\left(e^u f(u)+e^u\frac{df(u)}{du}\right)\Big|_{u=0}$$

$$=\frac{d^m}{du^m}e^u f(u)\Big|_{u=0}+b_m$$

is valid i.e. we have

$$a_{m+1}=a_m+b_m,$$

wherefrom it follows that

$$A\frac{df(u)}{du}=\Delta Af(u),$$

which proves 1° for $k=1$.

Let us suppose further that 1° is valid for $n\leqq k-1$, i.e. let us suppose that

$$A\frac{d^n f(u)}{du^n}=\Delta^n Af(u)\qquad (n=1,\ 2,\ \dots,\ k-1).$$

Since we have

$$a_{m+k}=\frac{d^{m+k}}{du^{m+k}}e^u f(u)\Big|_{u=0}=\frac{d^m}{du^m}\left(\frac{d^k}{du^k}e^u f(u)\right)\Big|_{u=0}$$

$$=\frac{d^m}{du^m}\left(\sum_{i=0}^{k}\binom{k}{i}f^{(i)}(u)\,e^u\right)\Big|_{u=0}$$

by using the induction hypothesis it follows that

$$a_{m+k}=a_m+\binom{k}{1}\Delta a_m+\cdots+\binom{k}{k-1}\Delta^{k-1}a_m+b_m^k.$$

On the other hand, since we have (see [1])

$$a_{m+k}=a_m+\binom{k}{1}\Delta a_m+\cdots+\binom{k}{k-1}\Delta^{k-1}a_m+\Delta^k a_m$$

it follows that $b_m^k=\Delta^k a_m$, i.e.

$$A\frac{d^k f(u)}{du^k}=(\Delta^k a_m)=\Delta^k Af(u),$$

which proves 1° for all n.

2° Let us write

$$\overset{s}{a}_m=\frac{d^m}{du^m}e^u\int_0^u f(t)\,dt\Big|_{u=0},$$

then we have

$$\overset{s}{a}_{m+1}=\frac{d^m}{du^m}\left(e^u\int_0^u f(t)\,dt+e^u f(u)\right)\Big|_{u=0}=\frac{d^m}{du^m}e^u\int_0^u f(t)\,dt\Big|_{u=0}+\frac{d^m}{du^m}e^u f(u)\Big|_{u=0}$$

$=\overset{s}{a}_m+a_m$, i.e.

$$\Delta\,\overset{s}{a}_m=a_m. \tag{4}$$

Since the general solution of the equation (4) is the sequence $\left(\sum_{i=1}^{m-1} a_i + C\right)$, where C is an arbitrary constant and $a_1^s = a_0$ it follows that

$$A \int_0^u f(t)\,\mathrm{d}t = \left(\sum_{i=0}^{m-1} a_i\right)$$

which proves 2°.

3° Let us suppose that $Au^k f^{(n)}(u) = (c_m)$. Then, on the basis of the definition we have

$$c_m = \frac{\mathrm{d}^m}{\mathrm{d}u^m} e^u u^k f^{(n)}(u)\Big|_{u=0} = \sum_{i=0}^{m} \binom{m}{i} \frac{\mathrm{d}^{m-i}}{\mathrm{d}u^{m-i}} u^k\Big|_{u=0} \frac{\mathrm{d}^i}{\mathrm{d}u^i} e^u f^{(n)}(u)\Big|_{u=0}, \text{ i.e.}$$

(5) $$c_m = \sum_{i=0}^{m} \binom{m}{i} \frac{\mathrm{d}^{m-i}}{\mathrm{d}u^{m-i}} u^k\Big|_{u=0} \Delta^n a_i.$$

Since the relation $\frac{\mathrm{d}^{m-i}}{\mathrm{d}u^{m-i}} u^k\Big|_{u=0} = k!$ is true only when $m-i=k$, and for $m-i \neq k$

$$\frac{\mathrm{d}^{m-i}}{\mathrm{d}u^{m-i}} u^k\Big|_{u=0} = 0,$$

from (5) it follows that

$$c_m = \binom{m}{m-k} k!\, \Delta^n a_{m-k}$$

i.e.

$$Au^k f^{(n)}(u) = (m^{(k)} \Delta^n a_{m-k})$$

which proves 3°.

4° On the basis of the definition it follows that

$$Au^k = \left(\frac{\mathrm{d}^m}{\mathrm{d}u^m} e^u u^k\Big|_{u=0}\right) = \left(\sum_{i=0}^{m} \binom{m}{i} \frac{\mathrm{d}^{m-i}}{\mathrm{d}u^{m-i}} e^u\Big|_{u=0} \frac{\mathrm{d}^i}{\mathrm{d}u^i} u^k\Big|_{u=0}\right).$$

On the other hand, as we proved, the relation $\frac{\mathrm{d}^i}{\mathrm{d}u^i} u^k\Big|_{u=0} = k!$ is valid only when $i=k$, while for $i \neq k$ we have $\frac{\mathrm{d}^i}{\mathrm{d}u^i} u^k\Big|_{u=0} = 0$; we get

$$Au^k = (m^{(k)})$$

which proves 4°.

By using the well known LEIBNIZ formula the general term of the sequence $Af(u)$ is given by

(6) $$a_m = \sum_{i=0}^{m} \binom{m}{i} f^{(i)}(0).$$

Theorem 4. *If the relation $Af(u) = Ag(u)$ is valid and if $f(u)$ and $g(u)$ can be developed in Maclaurin series then there exists a neighbourhood of zero in which the equality $f(u) = g(u)$ is true.*

Proof. Let us introduce the notation $Af(u)=(a_m)$, $Ag(u)=(b_m)$. On the basis of (6) for every $m \in \mathbf{N}_0$ we have

$$\left(\sum_{i=0}^{m}\binom{m}{i}f^{(i)}(0)\right)=\left(\sum_{i=0}^{m}\binom{m}{i}g^{(i)}(0)\right). \tag{7}$$

The equation (7) for $m=0, 1, 2, \ldots$ implies the following system of equalities $f^{(m)}(0)=g^{(m)}(0)\,(m \in \mathbf{N}_0)$.

On the basis of that fact, we can conclude that the functions $f(u)$ and $g(u)$ have the same Taylor expansion, so that those functions coincide in neighbourhood of zero.

3. Table

In this part of our paper we give a table of some elementary functions and their images by the A-transformation.

	$f(u)$	a_m
1	C (C — constant)	C
2	u^n	$m^{(n)}$
3	e^{au}	$(1+a)^m$
4	$\sin au$	$(1+a^2)^{m/2}\sin(m \operatorname{arctg} a)$
5	$\cos au$	$(1+a^2)^{m/2}\cos(m \operatorname{arctg} a)$
6	$u^n f(u)$	$m^{(n)}\,a_{m-n}$
7	$f_1(u)f_2(u)$	$\sum_{k=0}^{m}\binom{m}{k}a_k f_2^{(n-k)}(0)$ (where $Af_1(u)=(a_k)$).

4. Some applications of the A-transformation to difference equations

In this part we will give some applications of the A-transformation in solving some difference equations.

Example 1. By an application of the A-transformation to the equation

$$\frac{d^n y(x)}{dx^n}+b_1\frac{d^{n-1}y(x)}{dx^{n-1}}+\cdots+b_n y(x)=0 \tag{8}$$

where $b_i\,(i=1, \ldots, n)$ are real constants we get the equation

$$\Delta^n a_m+b_1\,\Delta^{n-1}a_m+\cdots+b_n a_m=0. \tag{9}$$

The general solution of the equation (8) is the sum of the following terms:

$$e^{rx},\ e^{(p+iq)x},\ x^n e^{rx},\ x^n e^{(p+iq)x} \qquad (n \in \mathbf{N}_0,\ p,\ q,\ r \in \mathbf{R}).$$

By application of the A-transformation we conclude that in the solution of (9) the above terms correspond to the following terms:

$$(1+r)^m,\ (1+p+iq)^m,\ m^{(n)}(1+r)^{m-n},\ m^{(n)}(1+p+iq)^{m-n}.$$

The general solution of the equation

$$y''+y'-y=0 \tag{10}$$

is given by

(11) $$y = C_1 \exp\left(\left(\frac{-1+\sqrt{5}}{2}\right)x\right) + C_2 \exp\left(\left(-\frac{1+\sqrt{5}}{2}\right)x\right)$$

where C_1 and C_2 are arbitrary constants.

By the application of the A-transformation to the equation (10) and the solution (11) it follows that the difference equation

$$\Delta^2 a_m + \Delta a_m - a_m = 0$$

i.e.

$$a_{m+2} = a_{m+1} + a_m$$

has the general solution

$$(a_m) = \left(C_1\left(\frac{1+\sqrt{5}}{2}\right)^m + C_2\left(\frac{1-\sqrt{5}}{2}\right)^m\right)$$

where C_1 and C_2 are arbitrary constants.

Example 2. The general solution of the equation

(12) $$\frac{d^n y(x)}{dx^n} = f(x)$$

is given by

(13) $$y(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt + C_{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots + C_0,$$

where $C_0, \ldots, C_{n-1}$ are arbitrary constants.

By the application of A-transformation to the equation (12) and the solution (13) it follows that the difference equation

$$\Delta^n a_m = b_m$$

has the solution

$$a_m = Ay(x).$$

By using the well known formulas:

$$(x+a)^{(i)} = \sum_{k=0}^{i} \binom{i}{k} x^{(k)} a^{(i-k)}, \quad (x+i)^{(i)} = (x+1)^{[i]}$$

and

$$(-x-a)^{(i)} = (-1)^i (x+a)^{[i]},$$

where $x^{[i]} = x(x+1)\cdots(x+i-1)$, the general term of the sequence $Ay(x) = (a_m)$ is given by

(14) $$a_m = \sum_{k=0}^{m-1} \frac{(m-k-1)^{(n-1)}}{(n-1)!} b_k + C_n m^{n-1} + \cdots + C_2 m + C_1$$

where $C_n, \ldots, C_1$ are arbitrary constants. The general solution (14) can be found in [2].

Example 3. The general solution of the equation

(15) $$\sum_{k=0}^{n} (-1)^k k! \binom{n}{k} x^{n-k} y^{(n-k)} = 0$$

is given by (see [3])

(16) $$y = \sum_{k=0}^{n} C_k x^k$$

where $C_0, \ldots, C_n$ are arbitrary constans.

By the application of the A-trransformation to the equation (15) and the solution (16) it follows that the difference equation

$$\sum_{k=0}^{n} (-1)^k k! \binom{n}{k} m^{(n-k)} \Delta^{n-k} a_{m-n+k} = 0$$

has the solution

$$\left(\sum_{i=0}^{n} C_i m^{(i)} \right)$$

where $C_0, \ldots, C_n$ are arbitrary constants.

EXAMPLE 4. Any particular solution of the equation

$$xy'' + (\alpha + 1 - x) y' + ny = 0 \qquad (\alpha \in \mathbf{R}) \tag{17}$$

is given by (see [4])

$$y(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}. \tag{18}$$

By the application of the A-transformation to the equation (17) and the any particular solution (18) it follows that the difference equation

$$(m + \alpha + 1) a_{m+1} - (3m + \alpha + 1 - n) a_m + 2 m a_{m-1} = 0$$

has any particular solution

$$(a_m) = \left(\sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \binom{m}{k} \right).$$

EXAMPLE 5. Any particular solution of the equation

$$x^2 y'' + xy' + (x^2 - n^2) y = 0 \qquad (n \in \mathbf{N}_0) \tag{19}$$

is given by (see [3])

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2} \right)^{n+2k}. \tag{20}$$

By the application of the A-transformation to the equation (19) and the particular solution (20) it follows that the difference equation

$$(m^2 - n^2) a_m - m(2m-1) a_{m-1} + 2m(m-1) a_{m-2} = 0$$

has any particular solution

$$(a_m) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{n+2k}\, k!\,(n+k)!} m^{(n+2k)}.$$

Professor B. CRSTICI and Professor D. BLANUŠA read the paper in manuscript and give very useful comments.

REFERENCES

1. L. BRAND: *Differential and difference equations*. New York—London—Sydney 1966.
2. A. GEL'FOND: *Isčislenie konečnyh raznostei*. Moskva 1967.
3. E. KAMKE: *Differentialgleichungen: Lösungsmethoden und Lösungen*. Leipzig 1959.
4. H. BATEMAN, A. ERDÉLYI: *Higher transcendental functions*, vol. 2. New York—Toronto—London 1953.