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670. ON SOME LINEAR TRANSFORMATIONS OF QUASI-MONOTONE SEQUENCES*

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In this paper a class of real sequences is considered which, in a special case, can be reduced to the class of nondcreasing sequences. It is proved that these classes are invariant with respect to some linear transformations.

1. Let $a=(a_n)(n=1, 2, \ldots)$ be a real sequence and let the sequence $A=(A_n)(n=1, 2, \ldots)$ be defined in the following way

$$A_n=\frac{p_1 a_1+\cdots+p_n a_n}{p_1+\cdots+p_n}, \tag{1}$$

where the weight sequence $p=(p_n)$ is strictly positive. Let K' and K'' be two classes of real sequences. In mathematical literature the implication

$$a\in K' \Rightarrow A\in K''$$

was, very often, considered where it is of interest to determine all the weights $p=(p_n)$ for which the above implication is valid. However, we must underline here, that it is plainly to consider this implication in the case when $K''=K'$. We have considered this implication, usually in the case when K' and K'' are the classes of convex sequences defined by the operators Δ^k (see, for example, our previous papers [1] and [2]). In these papers we have investigated the above implication only in the cases when $K'=K''$.

In the present paper we will consider also the same implication, but for this time, in the case when $K''\neq K'$, where the classes K' and K'' will be defined by some operators, which are analogous to the operator Δ and which in a special case reduces to the same operator. Namely, we will start with the following definition.

Definition 1. *Let $p\neq 0$ be a real constant. The operator L_p will be defined in the following way*

$$L_p(a_n)=a_{n+1}-pa_n \qquad (n\in \mathbf{N}). \tag{2}$$

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Clearly, this operator reduces to the operator Δ if $p=1$. We have limited ourself to the case $p \neq 0$, because if $p=0$ the above operator is not of interest with respect to the implication we have cosidered above.

In the definition which follows we will give a generalization of the concept of the monotony of real sequences.

Definition 2. *For a sequence $a=(a_n)$ we shall say that it is p-monotone or that it belongs to the class K_p if the inequality*

$$L_p(a_n) \geqq 0 \tag{3}$$

is valid for all $n \in \mathbf{N}$.

It is quite clear that the class K_p contains a sequence $a=(a_n)$ for which we have

$$L_p(a_n)=0 \tag{4}$$

for every $n \in \mathbf{N}$. The last equation has solution of the form

$$a_n = Cp^n \qquad (n=1, 2, \ldots), \tag{5}$$

where C is an arbitrary real constant.

In connection with operators L_p and corresponding classes K_p we will consider in the present paper for which weights $p=(p_n)$ the implication

$$a \in K_p \Rightarrow A \in K_q \tag{6}$$

holds true, where the sequence $A=(A_n)$ is defined by (1). As it will be shown in this paper of the special interest will be the case when we have $p \neq q$.

2. We will consider now the implication (6) in the form $a \in K_1 \Rightarrow A \in K_1$ i.e. when $p=q=1$. In that case the classes K_p and K_q are reduced to the class of nondecreasing sequences. It can be directly verified that the identity

$$A_{n+1}-A_n=\frac{p_1(a_{n+1}-a_1)+p_2(a_{n+1}-a_2)+\cdots+p_n(a_{n+1}-a_n)}{(p_1+p_2+\cdots+p_{n+1})(p_1+p_2+\cdots+p_n)}$$

holds true for arbitrary sequences $a=(a_n)$ and $p=(p_n)$ where the sequence $A=(A_n)$ is defined by (1). Therefrom we can easily conclude that the following statement is valid.

Lemma 1. *If the sequence $a=(a_n)$ is nondecreasing then the same property has the sequence $A=(A_n)$ where the weight sequence $p=(p_n)$ is an arbitrary positive sequence.*

However, the opposite statement can also be proved, i.e. we have:

Lemma 2. *Let for a given sequence $a=(a_n)$ the sequence $A=(A_n)$ be defined by* (1). *Then if for every sequence a and arbitrary positive weights $p=(p_n)$ the implication $a \in K_p \Rightarrow A \in K_p$ holds, then we must have $p=1$.*

Proof. From (5) it follows that two sequences (p^n) and $(-p^n)$ belong to the class K_p. By the application of the implication (6) (where we have taken $p=q$) to those two sequences we can conclude that

$$\frac{\sum_{k=1}^{n} p_k p^k}{\sum_{k=1}^{n} p_k} = Cp^n$$

where we have applied the definition (2) of the operator L_p and the corresponding relations (4) and (5). From the above equality (which holds for all $n \in \mathrm{N}$) for $n=1$ we obtain that $C=1$. From the same relation by putting $C=1$ and taking $n=2$ we get the following relation $p_1 p + p_2 p^2 = p^2(p_1+p_2)$ which, obviously holds, for arbitrary positive weights if and only if $p=1$. This proves the lemma 2.

Further on we will consider the implication (6) only for those values of p and q for which $p \neq q$. This situation is much more complicated then previous one. The necessary condition for the weight sequence $p=(p_n)$, for validity of the implication (6) for every sequence of the class K_p, is given in the following lemma.

Lemma 3. *Suppose that the sequence $A=(A_n)$, for a real sequence $a=(a_n)$, is given by* (1). *If the implication* (6) *is valid for every sequence $a=(a_n)$ then the weight sequence $p=(p_n)$ must be of the form*

$$p_n = p_1 \frac{q^{n-1}-q^{n-2}}{p^{n-1}-q^{n-2}} \prod_{k=1}^{n-1} \frac{p^k-q^{k-1}}{p^k-q^k} \qquad (n=2, 3, \ldots), \tag{7}$$

where the weight p_1 is arbitrary given positive number.

Proof. As we have said above the sequences (p^n) and $(-p^n)$ are in the class K_p. Since the implication (6) is valid for an arbitrary sequence from the class K_p, from this implication we have

$$L_q\left(\frac{\sum_{k=1}^{n} p_k p^k}{\sum_{k=1}^{n} p_k}\right) = 0 \qquad (n=1, 2, \ldots). \tag{8}$$

Using the fact that equation (4) has solution (5), from (8) it follows that

$$\frac{\sum_{k=1}^{n} p_k p^k}{\sum_{k=1}^{n} p_k} = Cq^n \qquad (n=1, 2, \ldots). \tag{9}$$

By putting $n=1$ in (9) we find $C=p/q$ (we have supposed that $p \neq 0$ and $q \neq 0$). Therefrom and from (9) we obtain the following relation

$$\sum_{k=1}^{n} p_k p^k = \frac{p}{q}\left(\sum_{k=1}^{n} p_k\right) q^n. \tag{10}$$

By subtracting the relations (10) for $n+1$ and n we find

$$(11)\qquad p_{n+1}p^{n+1}=\frac{p}{q}\left(q^{n+1}\sum_{k=1}^{n+1}p_k-q^n\sum_{k=1}^{n}p_k\right).$$

Denote

$$(12)\qquad B_n=\sum_{k=1}^{n}p_k\qquad (n=1,2,\ldots).$$

By using (12) the relation (11) can be written in the following form

$$(13)\qquad \frac{B_{n+1}}{B_n}=\frac{p^n-q^{n-1}}{p^n-q^n}=Q_n\qquad (n=1,2,\ldots).$$

By suitable multiplication of relations (13) for sequential values of n we have $B_n=B_1\prod_{k=1}^{n-1}Q_k$, which leads us to

$$(14)\qquad p_n=B_n-B_{n-1}=B_1(Q_{n-1}-1)\prod_{k=1}^{n-2}Q_k.$$

Using again the relations (12), (13) and (14) the above lemma 3 follows.

Lemma 4. *Suppose that real numbers p and q satisfy one of the following conditions*

$$(15)\qquad p>q>1,$$

$$(16)\qquad 0<p<q<1,$$

$$(17)\qquad p<q<0.$$

Then the weight sequence $p=(p_n)$ given by (7), where p_1 is arbitrary positive constant, is strictly positive.

Proof. The weight sequence, given by (7) is the product of the terms of the following form

$$\frac{q^{n-1}-q^{n-2}}{p^{n-1}-q^{n-2}}=\left(\frac{q}{p}\right)^{n-1}\frac{1-\dfrac{1}{q}}{1-\dfrac{1}{p}\left(\dfrac{q}{p}\right)^{n-2}},\qquad \frac{p^k-q^{k-1}}{p^k-q^k}=\frac{1-\dfrac{1}{p}\left(\dfrac{q}{p}\right)^{k-1}}{1-\left(\dfrac{q}{p}\right)^k}.$$

Since these terms are positive if one of the conditions (15), (16) or (17) is satisfied the proof of lemma 4 is finished.

The following lemma contains the sufficient conditions for the validity of the implication (6) in the case when $p\neq q$. In other words the following statement is valid.

Lemma 5. *Let us suppose that real numbers p and q satisfy the conditions of lemma 4. If the weight sequence $p=(p_n)$ is of the form (7), where p_1 is an arbitrary positive number, then the implication (6) is valid for an arbitrary sequence $a=(a_n)$ from the class K_p, where the sequence $A=(A_n)$ is defined by (1).*

Proof. We will determine the coefficients m_k $(k=1, \ldots, n)$ such that the identity

$$(18) \qquad L_q(A_n)=\sum_{k=1}^{n} m_k L_p(a_k)$$

holds true for an arbitrary $n \in \mathrm{N}$ and arbitrary sequence $a=(a_n)$. We have

$$(19) \qquad \sum_{k=1}^{n} m_k L_p(a_k)=-pm_1 a_1+\sum_{k=2}^{n}(m_{k-1}-pm_k)a_k+m_n a_{n+1}.$$

At the same time we have

$$(20) \qquad L_q(A_n)=\sum_{k=1}^{n}\left(\frac{p_k}{B_{n+1}}-q\frac{p_k}{B_n}\right)a_k+\frac{p_{n+1}}{B_{n+1}}a_{n+1}.$$

From (19) and (20) we find that

$$(21) \qquad -pm_1=p_1 L_q\left(\frac{1}{B_n}\right), \quad m_{k-1}-pm_k=p_k L_q\left(\frac{1}{B_n}\right) \ (k=2, \ldots, n), \quad m_n=\frac{p_{n+1}}{B_{n+1}}.$$

The system of conditions (21) implies that the coefficients m_k $(k=1, \ldots, n)$ must be of the form

$$(22) \qquad m_k=-\frac{S_n}{p^{k+1}}\sum_{j=1}^{k} p_j p^j \qquad (k=1, \ldots, n),$$

where we have introduced the sequence (S_n) by

$$(23) \qquad S_n=\frac{1}{B_{n+1}}-\frac{q}{B_n}.$$

The equality (10) in virtue of (12) can be writen in the following form

$$(24) \qquad \sum_{j=1}^{k} p_j p^j=\frac{p}{q}q^k B_k.$$

On the basis of (22), (23) and (24) we find that

$$(25) \qquad m_k=\frac{B_k}{B_n B_{n+1}}\frac{q^{k-1}}{p^k q^n}(q^{n+1}B_{n+1}-q^n B_n).$$

The relations (11) and (25) imply that the coefficients m_k are of the following form

$$(26) \qquad m_k=\frac{B_k}{B_n B_{n+1}}\left(\frac{p}{q}\right)^{n-k} p_{n+1} \qquad (k=1, \ldots, n).$$

Since the conditions (15), (16) and (17) are satisfied for p and q, from lemma 4 we have that $p_j>0$, i.e. from (26) it follows that $m_k \geqq 0$ $(k=1, \ldots, n)$. Since we have proved that the coefficients m_k are nonnegative the relation (18) proves that if the sequence $a=(a_n)$ is in the class K_p, then the sequence $A=(A_m)$ is in the class K_q. This completes the proof of lemma 5.

Previous lemmas 1—5 can be combined in the following theorem.

Theorem. *Let for a given sequence* $a=(a_n)$ *the sequence* $A=(A_n)$ *be dedined by* (1).

(i) *If we have* $p=q$ *then the implication* (6) *holds true for every sequence of the class* K_p *and for arbitrary positive weights* $p=(p_n)$ *if and only if* $p=q=1$.

(ii) *If* p *and satisfy one of the conditions* (15), (16) *or* (17) *then the implication* (6) *holds true for an arbitrary sequence of the class* K_p *if and only if the sequence* $p=(p_n)$ *of positive weights is given by* (7) *for* $n=2, 3, \ldots$ *where* p_1 *is an arbitrary positive number.*

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