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669. ON A LINEAR DIFFERENCE OPERATOR*

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This paper contains the proof that for the operator L defined by (2) the implication (4) holds for every sequence $x=(x_n)$ if and only if $L=\Delta^2$.

0. Let us denote by $x=(x_n)$ a sequence of real numbers, and let K be the class of real sequences convex of order $n\,(n=0, 1, 2, \ldots)$. Let, further, A be a given operator whose domian contains the class K and whose images are real sequences. It is interesting to consider for which classes K and operators A the implication

$$x \in K \Rightarrow A(x) \geqq 0 \tag{1}$$

holds true.

In today's literature there is a number of articles discussing the above question. Thus N. OZEKI [1] has proved the following theorem, which is in connection with the above implication (1).

Theorem 1. *For every real sequence $a=(a_n)$ convex of order* 2, *the sequence $A=(A_n)$, where A_n is defined by*

$$A_n=\frac{a_1+\cdots+a_n}{n},$$

is also convex of the same order.

P. M. VASIĆ, J. D. KEČKIĆ, I. B. LACKOVIĆ and Ž. M. MITROVIĆ had considered in paper [2] the same implication in the case when A is the second order difference of weighted arithmetic means and K is the class of all real sequences convex of order 2. In the same paper they obtained all the weights $p=(p_n)$ for which the implication (1) holds where A and K are defined as above.

Further generalizations of these results, for the sequences convex of order $n\,(n=3, 4, \ldots)$ and weighted arithmetic means, had been considered by I. B. LACKOVIĆ and S. K. SIMIĆ [3].

1. Denote by L the linear difference operator

$$L\,a_n=a_{n+2}+p\,a_{n+1}+q\,a_n, \qquad (n=1, 2, 3, \ldots) \tag{2}$$

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where $a=(a_n)$ is a given real sequence, p, q are given real numbers, and $q \neq 0$. Define, now, the sequence $\bar{x}=(\bar{x}_n)$ by

(3) $$\bar{x}_n=\frac{1}{n}\sum_{k=1}^{n} x_k.$$

Then, we shall consider for which p and q the implication

(4) $$L x_n \geqq 0 \Rightarrow L \bar{x}_n \geqq 0$$

holds, for every sequence $x=(x_n)$ and every $n \in \mathbf{N}$.

The difference equation

(5) $$L x_n=0,$$

indepedently of choice of p and q, has the solution of the form

(6) $$x_n=C_1 \alpha_n+C_2 \beta_n,$$

where C_1 and C_2 are arbitrary real constants.

In the following lemma, we shall give a necessary condition for the operator L so that the implication (4) holds true.

Lemma 1. *If for the operator L and every $x=(x_n)$ and every $n \in \mathbf{N}$ the implication* (4) *holds, then*

(7) $$L \bar{\alpha}_n=0,$$

and

(8) $$L \bar{\beta}_n=0$$

for everv $n \in \mathbf{N}$, where $\bar{\alpha}$ and $\bar{\beta}$ are defined with

(9) $$\bar{\alpha}_n=\frac{1}{n}\sum_{k=1}^{n} \alpha_k, \quad \bar{\beta}_n=\frac{1}{n}\sum_{k=1}^{n} \beta_k.$$

Proof. Consider the sequence $x=(x_n)$ defined by (6). This sequence satisfies (5), i.e. $L x_n \geqq 0$ for every $n \in \mathbf{N}$. Then we hawe $L \bar{x}_n \geqq 0$ from the implication (4), where

(10) $$\bar{x}_n=C_1 \bar{\alpha}_n+C_2 \bar{\beta}_n,$$

C_1, C_2 are arbitrary constants and the sequences $\bar{\alpha}$ and $\bar{\beta}$ are given by (9). Taking $C_1=+1$ and $C_2=0$ in (10), we have $L \bar{\alpha}_n \geqq 0$, but for $C_1=-1$, $C_2=0$ we have $-L \bar{\alpha}_n \geqq 0$, wherefrom we conclude that condition (7) for every $n \in \mathbf{N}$ must be fulfilled. In the similar way, by choosing $C_1=0$ and $C_2=\pm 1$, we derive (8). This completes the proof.

On the basis of lemma 1 we can make the next conclusion. If the implication (4) holds for the operator L and every sequence $x=(x_n)$, then the relations (7) and (8) follow, and next conditions must be fulfilled also

(11) $$\bar{\alpha}_3+p \bar{\alpha}_2+q \bar{\alpha}_1=0,$$

(12) $$\bar{\beta}_3+p \bar{\beta}_2+q \bar{\beta}_1=0.$$

Characteristic equation of equation (5), according to (2), is

$$t^2+pt+q=0. \tag{13}$$

We distinguish next three cases:

A) Let $p^2-4q>0$. Then (13) has the form $t^2-(t_1+t_2)t+t_1t_2=0$, where t_1, t_2 are real numbers such that $t_1\cdot t_2\neq 0$ and $t_1\neq t_2$. We have $\alpha_n=t_1^n$, $\beta_n=t_2^n$ and

$$\bar{\alpha}_n=\frac{1}{n}\sum_{k=1}^{n}t_1^k,\quad \bar{\beta}_n=\frac{1}{n}\sum_{k=1}^{n}t_2^k.$$

Now (11) and (12) have the forms

$$\frac{t_1+t_1^2+t_1^3}{3}-(t_1+t_2)\frac{t_1+t_1^2}{2}+t_1t_2\cdot t_1=0, \tag{14}$$

$$\frac{t_2+t_2^2+t_2^3}{3}-(t_1+t_2)\frac{t_2+t_2^2}{2}+t_1t_2\cdot t_2=0, \tag{15}$$

and using the condition $q\neq 0$, i.e. $t_1t_2\neq 0$ we have

$$2-t_1-t_1^2+3t_1t_2-3t_2=0, \tag{16}$$

$$2-t_2-t_2^2+3t_1t_2-3t_1=0. \tag{17}$$

Subtracting the last two equations we derive $(t_2-t_1)(t_1+t_2-2)=0$.

As $t_1\neq t_2$ it must be $t_2=2-t_1$. Then, on the basis of (16) we have $(t_1-1)^2=0$, therefrom it follows $t_1=t_2$.

The result we just derived, can be formulated in following way:

Lemma 2. *Let* L *be defined by* (2) *where* $q\neq 0$ *and* $p^2-4q>0$. *Then the implication* (4) *cannot be true for every* $x=(x_n)$ *and for every* $n\in\mathbf{N}$.

B) Now, $p^2-4q=0$. Then (13) may be written as $(t-t_0)^2=0$ where $t_0\in\mathbf{R}$ and $t_0\neq 0$. Now we have $\alpha_n=t_0^n$, $\beta_n=nt_0^n$ i.e.

$$\bar{\alpha}_n=\frac{1}{n}\sum_{k=1}^{n}t_0^k;\quad \bar{\beta}_n=\frac{1}{n}\sum_{k=1}^{n}kt_0^k.$$

Equalities (11) and (12) have the forms

$$\frac{t_0+t_0^2+t_0^3}{3}-2t_0\frac{t_0+t_0^2}{2}+t_0^2t_0=0, \tag{18}$$

$$\frac{t_0+2t_0^2+3t_0^3}{3}-2t_0\frac{t_0+2t_0^2}{2}+t_0^2t_0=0. \tag{19}$$

(18) and (19) get the forms

$$(1-t_0)^2=0, \tag{20}$$

$$1-t_0=0 \tag{21}$$

after evident transformations. In this way we obtain the next result:

Lemma 3. *Let L be defined by* (2) *where $q \neq 0$ and $p^2 - 4q = 0$. Then, if the implication* (4) *holds for every $x = (x_n)$ and every $n \in \mathbf{N}$, it must be $p = -2$ and $q = 1$.*

C) $p^2 - 4q < 0$. Then (13) has a solution $t_{1,2} = \rho(\cos\omega \pm i \sin\omega)$ i.e. (13) gets the form

$$(t - \rho e^{i\omega})(t - \rho e^{-i\omega}) = t^2 - 2t\rho\cos\omega + \rho^2,$$

wherefrom it follows $p = -2\rho\cos\omega$, $q = \rho^2$. Now we have

$$\alpha_n = \rho^n \cos n\omega, \quad \beta_n = \rho^n \sin n\omega,$$

i.e.

$$\bar{\alpha}_n = \frac{1}{n}\sum_{k=1}^{n} \rho^k \cos k\omega; \quad \bar{\beta}_n = \frac{1}{n}\sum_{k=1}^{n} \rho^k \sin k\omega.$$

In this manner (11) and (12) have the form

$$\text{(22)} \qquad \frac{\rho\cos\omega + \rho^2\cos 2\omega + \rho^3\cos 3\omega}{3} - 2\rho\cos\omega\,\frac{\rho\cos\omega + \rho^2\cos 2\omega}{2} + \rho^2\rho\cos\omega = 0,$$

$$\text{(23)} \qquad \frac{\rho\sin\omega + \rho^2\sin 2\omega + \rho^3\sin 3\omega}{3} - 2\rho\cos\omega\,\frac{\rho\sin\omega + \rho^2\sin 2\omega}{2} + \rho^2\rho\sin\omega = 0.$$

From the relations (22) and (23) we obtain

$$\text{(24)} \qquad \frac{1}{3}(z + z^2 + z^3) - \frac{1}{2}(z + \bar{z})(z + z^2) + z\bar{z}z = 0,$$

where we introduced the substitution $z = \rho e^{i\omega}$.

On the basis of the asumption $q \neq 0$ we have $\rho \neq 0$, i.e. $z \neq 0$ so (24) becomes

$$\text{(25)} \qquad 2(1 + z + z^2) - 3(1 + z)(z + \bar{z}) + 6z\bar{z} = 0.$$

This equation can be writen in the form

$$\text{(26)} \qquad 2(1 + \bar{z} + \bar{z}^2) - 3(1 + \bar{z})(z + \bar{z}) + 6z\bar{z} = 0.$$

Substracting (25) from (26) we have $2(z - \bar{z}) + 2(z^2 - \bar{z}^2) - 3(z + \bar{z})(z - \bar{z}) = 0$, i.e.

$$\text{(27)} \qquad (z - \bar{z})(2 - z - \bar{z}) = 0.$$

Solutions of (27) are

$$\text{(28)} \qquad z = \bar{z},$$

$$\text{(29)} \qquad \bar{z} = 2 - z.$$

It is clear that the condition (28) cannot be fulfilled because of $p^2 - 4q < 0$. Substituting (29) into (25), we have

$$\text{(30)} \qquad -4(z - 1)^2 = 0$$

which cannot be fulfilled too.

In this manner the following lemma has been proved.

Lemma 4. *Let the operator L be defined by* (2), *where $q \neq 0$ and $p^2 - 4q < 0$. Then the inplication* (4) *cannot be valid for every sequence $x = (x_n)$ and every $n \in \mathbf{N}$.*

On the basis of the proved lemmas and theorem 1 we have:

Theorem 2. *Let the operator L have the form defined by* (2), *where $p, q \in \mathbf{R}$, $q \neq 0$. Then the implication* (4) *holds for every sequence $x = (x_n)$ if and only if $p = -2$, $q = 1$, i.e. if and only if $L \equiv \Delta^2$.*

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