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668. INTERPOLATION IN THE $L^{p,\lambda}$-SPACES AND ELLIPTIC DIFFERENTIAL EQUATIONS*

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1. In this paper we prove the interpolation theorem for a special case of spaces $L^{p,\lambda}$ with mixed norms. Then taking advantage of some properties of $L^{p,\lambda}$-spaces and the above theorem, proven heretofore, their application to the theory of partial differential equations of elliptic type is demonstrated. Results are of a similar type as those published before in [3] and [4]. Feasibility of such studies has been discussed in [7] 4.11.

2. The index $i=1, \ldots, n$, unless otherwise stated. Let $\mathbf{R}$ be the set of real numbers, k_i – positive integer, $\sum_{i=1}^{n} k_i = N$, $1 \leqq p_i, r_i, s_i, q_i \leqq \infty$, $\lambda_i \geqq 0$. In the following we shall apply vector notations, i. e. $p=(p_1, \ldots, p_n)$, $x=(x_1, \ldots, x_n)$ etc. Let $Q_i^0(x_i^0, r_i) \subset \mathbf{R}^{k_i}$ be the fixed, bounded cube with centre at x_i^0 and edge length equal to r_i, the edges of which are parallel to coordinate axes. We shall denote by Q_i the subcube of Q_i^0 and homothetic with Q_i^0. Let Ω_i denote an open bounded subset of $\mathbf{R}^{k_i}$. Assume $\Omega_i(x_i^0, r_i) = \Omega_i \cap Q_i^0(x_i^0) r_i)$,

$$Q^0 = P_{i=1}^{n} Q_i^0,\ Q = P_{i=1}^{n} Q_i,\ \Omega = P_{i=1}^{n} \Omega_i,\ \bar{\Omega} = P_{i=1}^{n} \bar{\Omega}_i,\ \Omega(x^0, r) = P_{i=1}^{n} \Omega_i(x_i^0, r_i).$$

The measure means always Lebesque measure. To simplify the notation, we shall write, for example:

$$\int_{\Omega} u(x)\,\mathrm{d}x = \int_{\Omega_n} \cdots \int_{\Omega_1} u(x)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

$$\int_{b\Omega} |u(x)|^p\,\mathrm{d}x = \|u\|_{L^p(\Omega)}^{p_n} = \int_{\Omega_n} \Big[\cdots \int_{\Omega_2} \Big(\int_{\Omega_1} |u(x)|^{p_1}\,\mathrm{d}x_1 \Big)^{p_2/p_1} \mathrm{d}x_2 \cdots \Big]^{p_n/p_{n-1}} \mathrm{d}x_n.$$

Definition 1. *We shall denote by $L^{p,\lambda}(\Omega)$ the linear space of functions u locally integrable in Ω for which there exists a positive constant M depending on u such that for every x^0, r with $x_i^0 \in \Omega_i$, $r_i > 0$ there holds the inequality*

$$\Big\{ \int_{b\Omega(x_0, r)} \big| u(x) - u_{\Omega(x^0, r)} \big|^p\,\mathrm{d}x \Big\}^{1/p_n} \leqq M \prod_{i=1}^{n} r_i^{\lambda_i p_n / p_i}$$

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where

$$u_{\Omega(x^0, r)} = \{P_{i=1}^{n} \mu [\Omega_i(x_i^0, r_i)]\}^{-1} \int_{\Omega(x^0, r)} u(x)\,dx$$

is the mean value of u *on* $\Omega(x^0, r)$.
The expression

$$|||u|||_{L^{p,\lambda}(\Omega)} \left\{ \sup_{\substack{x_i^0 \in \Omega_i \\ r_i > 0}} \prod_{i=1}^{n} r_i^{-\lambda_i p_n / p_i} \int_{b\Omega(x^0, r)} |u(x) - u_{\Omega(x^0, r)}|^p \,dx \right\}^{1/p_n}$$

is a seminorm in $L^{p,\lambda}(\Omega)$.

The space $L^{p,\lambda}(\Omega)$ is a norm space with, for instance, the following norm

$$\|f\|_{L^{p,\lambda}(\Omega)} = \|f\|_{L^p(\Omega)} + |||f|||_{L^{p,\lambda}(\Omega)}.$$

Definition 2. *We shall denote by* $\varepsilon_0(Q^0)$ *the linear space of functions* u *locally integrable in* Q^0, *for which there exists a positive constant* M *depending on* u *such that for every* $Q_i \subset Q_i^0$ *there holds the inequality*

$$\int_Q |u(x) - u_Q| \,dx \leq M \prod_{i=1}^{n} \mu(Q_i).$$

The expression

$$|||u|||_{\varepsilon_0(Q^0)} = \sup_{Q_i \subset Q_i^0} \prod_{i=1}^{n} [\mu(Q_i)]^{-1} \int_Q |u(x) - u_Q| \,dx$$

is a seminorm in $\varepsilon_0(Q^0)$.

The norm is, for instance,

$$\|u\|_{\varepsilon_0(Q^0)} = |||u|||_{\varepsilon_0(Q^0)} + \|u\|_{L^1(Q^0)}.$$

Let T be a linear mapping defined on $L^1(Q^0)$.

Theorem 1. *Let the linear mapping* T *be continuous simultaneously from* $L^\infty(\Omega)$ *into* $\varepsilon_0(Q^0)$ *and from* $L^p(\Omega)$ *into* $L^p(Q^0)$ *such that there hold the inequalities*

(1) $$|||Tu|||_{\varepsilon_0(Q^0)} \leq M_1 \|u\|_{L^\infty(\Omega)}, \quad \forall u \in L^\infty(\Omega),$$

(2) $$\|Tu\|_{L^p(Q^0)} \leq M_2 \|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega), \; p_n \geq \cdots \geq p_1 \geq 1.$$

Then, for all $t \in [0, 1]$, T *is a continuous linear mapping from* $L^r(\Omega)$ *into* $L^s(Q^0)$, *where*

(3) $$\frac{1}{r_i} = \frac{1-t}{q_i} + \frac{t}{p_i}, \quad \frac{1}{s_i} = \frac{1-t}{1} + \frac{t}{p_i}, \quad p_i \leq q_i,$$

and for every $u \in L^r(\Omega)$ *the inequality*

(4) $$\|Tu\|_{L^s(Q^0)} \leq C_1 \|u\|_{L^r(\Omega)}$$

holds, where C_1 *is a constant depending on* m, p, q, r, s, Ω, Q^0, M_1, M_2.

Proof. Let $\Delta : Q^0 = P_{i=1}^{n} Q_i^0 = \bigcup_k P_{i=1}^{n} Q_{ik}$, Q_{ik} — cube contained in Q_i, be a decomposition of Q^0 into a denumerable number of products $P_{i=1}^{n} Q_{ik}$, notwo of which have a common interior point. For every $u \in L^p(\Omega)$ let us denote by $\tau(u)$ the function defined on Q^0 which for every product $P_{i=1}^n Q_{ik}$ of decomposition Δ has the constant value

$$\prod_{i=1}^{n} [\mu(Q_{ik})]^{-1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu - (Tu)_{P_{i=1}^n Q_{ik}}| \, dx$$

i. e.

$$(\tau u)(t) = \sum_k \prod_{i=1}^{n} [\mu(Q_{ik})]^{-1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu - (Tu)_{P_{i=1}^n Q_{ik}}| \, dx \, \chi_{P_{i=1}^n Q_{ik}}(t).$$

We observe that the mapping τ is sub-linear and

$$\| \tau u \|_{L^\infty(Q^0)} \leq ||| Tu |||_{\varepsilon_0(Q^0)}.$$

Hence by (1) we have

$$\| \tau u \|_{L^\infty(Q^0)} \leq M_1 \| u \|_{L^\infty(\Omega)}. \tag{5}$$

Let us notice that for $p_n \geq \cdots \geq p_2 \geq p_1 \geq 1$ we have

$$\| \tau u \|_{L^p(Q^0)} = \Big\{ \int\limits_{bQ^0} |(\tau u)(t)|^p \, dt \Big\}^{1/p_n}$$

$$\leq 2^{\frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}}} \sum_k \prod_{i=1}^{n} [\mu(Q_{ik})]^{-1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu - (Tu)_{P_{i=1}^n Q_{ik}}| \, dx \, \| \chi_{P_{i=1}^n Q_{ik}} \|_{L^p(Q^0)}$$

$$\leq 2^{\frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}}} \sum_k \prod_{i=1}^{n} [\mu(Q_{ik})]^{\frac{1}{p_i} - 1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu - (Tu)_{P_{i=1}^n Q_{ik}}| \, dx$$

$$\leq 2^{\frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}} + 1} \prod_{i=1}^{n} [\mu(Q_i^0)]^{\frac{1}{p_i} - 1} \int\limits_{Q^0} |Tu| \, dx.$$

Hence applying HÖLDER's inequality and (2) we get

$$\| \tau u \|_{L^p(Q^0)} \leq C_2(p, M_2) \| u \|_{L^p(\Omega)}. \tag{6}$$

Thus by **[5]** n.3 (see also **[6]**) we obtain

$$\| \tau u \|_{M^p(Q^0)} \leq C_2(p, M_2) \| u \|_{L^p(\Omega)}. \tag{7}$$

Then, the sub-linear mapping τ is continuous from $L^\infty(\Omega)$ into $\varepsilon_0(Q^0)$ and from $L^p(\Omega)$ into $M^p(Q^0)$ and the inequalities (5) and (7) hold i. e. τ is of weak types: (∞, ∞) and (p, p). Applying Th. 1 **[6]** we obtain that τ is of a strong type (q, q) and for every $p_i < q_i < \infty$ the following inequality holds

$$\| \tau u \|_{L^q(Q^0)} \leq C_3(q, q) \, M_1^{1-(p_i/q_i)} \, C_2^{p_i/q_i}(p, M_2) \| u \|_{L^q(\Omega)}.$$

Hence

(8) $$\|\tau u\|_{L^q(Q^0)} \leqq C_4(p, q, M_1, M_2)\,\|u\|_{L^q(\Omega)}.$$

Next, let us prove that the linear mapping $u \mapsto [Tu-(Tu)_{Q^0}]$ is:

a) of strong type $(q, 1)$, b) of strong type (p, p)

and there hold the inequalities

(9) $$\|Tu-(Tu)_{Q^0}\|_{L^1(Q^0)} \leqq C_5(p, q, Q^0, M_1, M_2)\,\|u\|_{L^q(\Omega)},$$

(10) $$\|Tu-(Tu)_{Q^0}\|_{L^p(Q^0)} \leqq 2\,M_2\,\|u\|_{L^p(\Omega)}.$$

Proof of a). We have for $q_n \geqq \cdots \geqq q_2 \geqq q_1 \geqq 1$

$$\sup_{\Delta} \sum_k \prod_{i=1}^n [\mu(Q_{ik})]^{(1/q_i)-1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu-(Tu)_{P_{i=1}^n Q_{ik}}|\,dx$$

$$= \sup_{\Delta} \sum_k \prod_{i=1}^n [\mu(Q_{ik})]^{-1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu-(Tu)_{P_{i=1}^n Q_{ik}}|\,dx\,\|\chi_{P_{i=1}^n Q_{ik}}\|_{L^q(Q^0)}$$

$$\leqq \|\tau u\|_{L^q(Q^0)}.$$

If we take a decomposition Δ of the set $P_{i=1}^n Q_i^0$ containing only one set, i.e. $\bigcup_k P_{i=1}^n Q_{ik} = P_{i=1}^n Q_i^0$ then we have

$$\prod_{i=1}^n [\mu(Q_i^0)]^{(1/q_i)-1} \int\limits_{P_{i=1}^n Q_i^0} |Tu-(Tu)_{P_{i=1}^n Q_i^0}|\,dx$$

$$\leqq \sup_{\Delta} \sum_k \prod_{i=1}^n [\mu(Q_{ik})]^{(1/q_i)-1} \int\limits_{P_{i=1}^n Q_{ik}} |Tu-(Tu)_{P_{i=1}^n Q_{ik}}|\,dx.$$

We obtain from the two above inequalities

$$\|Tu-(Tu)_{Q^0}\|_{L^1(Q^0)} \leqq \prod_{i=1}^n [\mu(Q_i^0)]^{1-(1/q_i)}\,\|\tau u\|_{L^q(Q^0)}.$$

Hence and by (8) we get (9).

Proof of b). If we apply MINKOWSKI inequality, HÖLDER's inequality and (2), successively, we get

$$\|Tu-(Tu)_{Q^0}\|_{L^p(Q^0)} \leqq \|Tu\|_{L^p(Q^0)} + |(Tu)_{Q^0}| \prod_{i=1}^n [\mu(Q_i^0)]^{1/p_i}$$

$$\leqq 2\,M_2\,\|u\|_{L^p(Q^0)}.$$

Hence (10) follows.

Then, by (9) and (10), we apply the RIESZ-THORIN th. [1] and get that the linear mapping $u \mapsto [Tu-(Tu)_{Q^0}]$ is of strong type (r, s) where the conditions (3) are satisfied and the following inequality holds

(11) $$\|Tu-(Tu)_{Q^0}\|_{L^s(Q^0)} \leqq C_5^{1-t}(2\,M_2)^t\,\|u\|_{L^r(\Omega)} = C_6(p, q, r, s, Q^0, M_1, M_2)\,\|u\|_{L^r(\Omega)}.$$

Taking into account MINKOWSKI's inequality, HÖLDER's inequality, (2) and (11) we obtain for $r_i \geqq p_i$

$$\| Tu \|_{L^s(Q^0)} \leqq \| Tu - (Tu)_{Q^0} \|_{L^s(Q^0)} + |(Tu)_{Q^0}| \prod_{i=1}^{n} [\mu(Q_i^0)]^{1/s_i}$$

$$\leqq C_5 \| u \|_{L^r(\Omega)} + \prod_{i=1}^{n} [\mu(Q_i^0)]^{(1/s_i)-(1/p_i)} \| Tu \|_{L^p(Q^0)}$$

$$\leqq C_5 \| u \|_{L^r(\Omega)} + \prod_{i=1}^{n} [\mu(Q_i^0)]^{(1/s_i)-(1/p_i)} \cdot M_2 \| u \|_{L^p(\Omega)}$$

$$\leqq C_5 \| u \|_{L^r(\Omega)} + \prod_{i=1}^{n} [\mu(Q_i^0)]^{(1/s_i)-(1/p_i)} \cdot [\mu(\Omega_i)]^{(1/p_i)-(1/r_i)} \cdot M_2 \| u \|_{L^r(\Omega)}.$$

Hence (4) follows.

From Def. 1 and Def. 2 we notice easily that

$$||| u |||_{\varepsilon_0(Q^0)} = ||| u |||_{L^{1,N}(Q^0)}$$

and that the spaces $L^{p,0}(\Omega)$ and $L^p(\Omega)$ are isomorphic. Also we know that bilipschitz transformation leaves $L^{p,\lambda}(\Omega)$ invariant (see **[4]** n. 1). Hence, we can substitute the respective norms and seminorms in Th. 1 by equivalent norms and seminorms, formally writing Ω instead of Q^0. Then we obtain

Theorem 2. *Let the linear mapping T be continuous simultaneously from $L^\infty(\Omega)$ into $L^{1,N}(\Omega)$ and from $L^p(\Omega)$ into $L^p(\Omega)$ and such that there hold the inequalities*

(1)′ $\quad ||| Tu |||_{L^{1,N}(\Omega)} \leqq M_1 \| u \|_{L^\infty(\Omega)}, \ \forall u \in L^\infty(\Omega),$

(2)′ $\quad \| Tu \|_{L^p(\Omega)} \leqq M_2 \| u \|_{L^p(\Omega)}, \ \forall u \in L^p(\Omega), \ p_n \geqq \dots \geqq p_1 \geqq 1.$

Then for all $t \in [0, 1]$ T is continuous linear mapping from $L^r(\Omega)$ into $L^s(\Omega)$ (where conditions (3) *are satisfied) and for every $u \in L^r(\Omega)$ the inequality*

(4)′ $\quad \| Tu \|_{L^s(\Omega)} \leqq C_1 \| u \|_{L^r(\Omega)}$

holds.

3. Basic notions of this part are taken from **[4]** and **[7]**. Let us consider a linear differential elliptic operator of the second order:

$$E(u) = \sum_{i,j=1}^{N} \frac{\partial}{x_j} \left[a_{ij}(x) \frac{\partial u}{\partial x_i} \right], \qquad a_{ij} = a_{ji}$$

where a_{ij} satisfy HÖLDER's condition in $\overline{\Omega}$ and following condition is satisfied:

$$\nu^{-1} |\xi|^2 \leqq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leqq \nu |\xi|^2, \qquad \nu > 0$$

for every $\xi \in \mathbf{R}^N$, $x \in \overline{\Omega}$.

Let $f_j \in L^2(\Omega)$, $j = 1, \dots, N$. Let $u \in H_0^{1,2}(\Omega)$ (where $H_0^{1,2}(\Omega)$ be the set of functions u with compact support in Ω, such that $u \in L^2(\Omega)$ and its distributional derivatives $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, $i = 1, \dots, N$). Moreover, let u be a solution of the equation

$$E(u) = \sum_{j=1}^{N} \frac{\partial f_i}{\partial x_j}.$$

It is well known (see [4] n. 3) that a unit solution exists and that

$$\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\Omega)} \leqq \nu^2 \sum_{j=1}^{N} \|f_j\|^2_{L^2(\Omega)}. \tag{12}$$

It was proved by S. Campanato, see [2], that if $f_j \in L^{2,N}(\Omega)$ and is of the class $C^{1,\alpha}$, $\alpha > 0$ (for def. of $C^{1,\alpha}$ see [2] p. 362) then there exists a constant $C_7 > 0$ such that

$$\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{2,N}(\Omega)} \leqq C_7 \sum_{j=1}^{N} \|f_j\|_{L^{2,N}(\Omega)}. \tag{13}$$

On the other hand we know by Def. 1 and by Hölder's inequality, that

$$\|u\|_{L^{1,N}(\Omega)} \leqq C_8 \|u\|_{L^{2,N}(\Omega)}. \tag{14}$$

We know also by [7]- 4.7 that

$$\|u\|_{L^{2,N}(\Omega)} \leqq C_9 \|u\|_{L^\infty(\Omega)}. \tag{15}$$

The inequalities (13), (14) and (15) imply

$$\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{1,N}(\Omega)} \leqq C_{10} \sum_{j=1}^{N} \|f_j\|_{L^\infty(\Omega)}. \tag{16}$$

Now, let G be the Green's operator corresponding to the Dirichlet problem described above. Putting $F=(f_1, \dots, f_N)$, we can write $u=G(F)$. Denote by G_{hk} the linear operators defined on $L^2(\Omega)$ by $G_{hk}(f)=\frac{\partial}{\partial x_h} G(F_k)$ where $F_k = (0, \dots, 0, f, 0, \dots, 0)$ with f in the k-th place. It is obvious that $\frac{\partial u}{\partial x_h} = \sum_{k=1}^{N} G_{hk}(f_k)$. Estimates (12) and (16) indicate that every G_{hk} is continuous linear operator from $L^2(\Omega)$ into $L^2(\Omega)$ as well from $L^\infty(\Omega)$ into $L^{1,N}(\Omega)$. Then (1)′ and (2)′ for $p_i=2$ of Th. 2 are satisfied. Thus G_{hk} is continuous inear operator from $L^r(\Omega)$ into $L^s(\Omega)$, with conditions (3), ($h, k=1, \dots, N$) and

$$\sum_{n=1}^{N}\left\|\frac{\partial u}{\partial x_h}\right\|_{L^s(\Omega)} \leqq C_{11} \sum_{n=1}^{N} \|fh\|_{L^r(\Omega)}.$$

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