UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 57—61.

642. SOME ESTIMATES OF L^r NORM ON THE SET OF CONTINUOUSLY-DIFFERENTIABLE FUNCTIONS

Radosav Ž. Đorđević, Gradimir V. Milovanović and *Josip E. Pečarić*

1. In [1] the following generalization of the Theorem of V. A. ZMOROVIČ ([2]) is proved:

Theorem A. *Let the function $f\colon [a-h,\ a+h]\to \mathbf{R}$ be twice continuously-differentiable and $p\colon [a-h,\ a+h]\to \mathbf{R}^+$ continuous.*

Then

$$\int_{a-h}^{a+h} p(x)\,|f''(x)|^r\,\mathrm{d}x \geqq \frac{h^{2r}}{q(r)^{r-1}}\,|\Delta|^r \qquad (r>1), \tag{1}$$

where

$$q(r)=\int_0^h (h-t)^{\frac{r}{r-1}}\left(p(a-t)^{\frac{1}{1-r}}+p(a+t)^{\frac{1}{1-r}}\right)\mathrm{d}t$$

and

$$\Delta=\frac{1}{h^2}\,\big(f(a+h)-2f(a)+f(a-h)\big).$$

Equality in (1) *holds if and only if the function f is given by*

$$f(x)=\begin{cases} A_1\displaystyle\int_a^x (x-t)\left[\frac{h-a+t}{p(t)}\right]^{\frac{1}{r-1}}\mathrm{d}t+A_2x+A_3 & (x\in[a-h,\ a]),\\[2ex] A_1\displaystyle\int_a^x (x-t)\left[\frac{h+a-t}{p(t)}\right]^{\frac{1}{r-1}}\mathrm{d}t+A_2x+A_3 & (x\in[a,\ a+h]),\end{cases}$$

where A_1, A_2, A_3 are arbitrary constants.

In this paper we will use standard operators D (differentiation operator), δ (central difference operator), μ (averaging operator), which are defined by

$$Df(x)=h\frac{\mathrm{d}f(x)}{\mathrm{d}x},\qquad \delta f(x)=f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right),$$

$$\mu f(x)=\frac{1}{2}\left(f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right).$$

If we put

$$\|\Phi\|_{r,p}=\left(\frac{1}{2h}\int_{a-h}^{a+h} p(x)\,|\Phi(x)|^r\,dx\right)^{1/r},\quad \|\Phi\|_r=\|\Phi\|_{r,1}\qquad (r>1),$$

$$\sigma_n(r)=\left(\frac{1}{2h}\int_0^h (h-t)^{\frac{(n-1)r}{r-1}}\left(p(a-t)^{\frac{1}{1-r}}+p(a+t)^{\frac{1}{1-r}}\right)dt\right)^{\frac{r-1}{r}},$$

the inequality (1) can be written in the form

$$\|f''\|_{r,p}\geqq\frac{1}{2h\,\sigma_2(r)}\,|\delta^2 f(a)|.$$

2. First we will give in this paper, one natural generalization of the last inequality.

Theorem 1. *Let $f\in C^n[a-h,a+h]$ and the function p: $[a-h,a+h]\to \mathbf{R}^+$ is continuous. Then*

$$\|f^{(n)}\|_{r,p}\geqq\frac{(n-1)!}{h\,\sigma_n(r)}\,|M_n f(a)|\qquad (r>1),\tag{2}$$

where

$$M_n=\frac{1}{2}\delta^2-\sum_{i=1}^{\frac{n-2}{2}}\frac{D^{2i}}{(2i)!}\qquad (n\ \text{is even}),\tag{3}$$

$$=\mu\delta-\sum_{i=1}^{\frac{n-1}{2}}\frac{D^{2i-1}}{(2i-1)!}\qquad (n\ \text{is odd}).$$

Equality in (2) *holds if and only if the function f is given by*

$$f(x)=\begin{cases} A_n\displaystyle\int_a^x (x-t)^{n-1}\left[\frac{(h-a+t)^{n-1}}{p(t)}\right]^{\frac{1}{r-1}}dt+\sum_{k=0}^{n-1}A_k(x-a)^k & (x\in[a-h,a]),\\[2ex] A_n\displaystyle\int_a^x (x-t)^{n-1}\left[\frac{(h+a-t)^{n-1}}{p(t)}\right]^{\frac{1}{r-1}}dt+\sum_{k=0}^{n-1}A_k(x-a)^k & (x\in[a,a+h]),\end{cases}\tag{4}$$

where A_k $(k=0,1,\ldots,n)$ are arbitrary real constants.

Proof. Similarly as in the paper [1], we find out (for $r>1$)

$$\frac{1}{2h}\int_{a-h}^{a+h} p(x)\,|f^{(n)}(x)|^r\,dx\tag{5}$$

$$\geqq\frac{1}{2h}\int_0^h\left(p(a-t)^{\frac{1}{1-r}}+p(a+t)^{\frac{1}{1-r}}\right)^{1-r}|f^{(n)}(a-t)+f^{(n)}(a+t)|^r\,dt$$

and

$$\frac{1}{2h}\int_0^h (h-t)^{n-1}\,|f^{(n)}(a-t)+f^{(n)}(a+t)|\,\mathrm{d}t \leqq \sigma_n(r)\,P^{1/r},$$

where the left side of the inequality (5) is asigned with P.

Since

$$R_n=\int_0^h (h-t)^{n-1}\left(f^{(n)}(a-t)+f^{(n)}(a+t)\right)\mathrm{d}t$$

$$=(n-1)(n-2)R_{n-2}-2(n-1)h^{n-2}f^{(n-2)}(a),$$

i. e.,

$$R_{2k}=(2k-1)!\left\{f(a+h)-2f(a)+f(a-h)-2\sum_{i=1}^{k-1}\frac{h^{2i}}{(2i)!}f^{(2i)}(a)\right\}$$

and

$$R_{2k+1}=(2k)!\left\{f(a+h)-f(a-h)-2\sum_{i=1}^{k}\frac{h^{2i-1}}{(2i-1)!}f^{(2i-1)}(a)\right\},$$

we have

$$R_n=2(n-1)!\,M_nf(a),$$

where the operator M_n is defined by (3).

On the basis of the above, we conclude that the inequality

$$\|f^{(n)}\|_{r,p}\geqq\frac{1}{\sigma_n(r)}\frac{1}{2h}|R_n|=\frac{(n-1)!}{h\,\sigma_n(r)}|M_nf(a)| \qquad (r>1)$$

holds, with equality if and only if f is given by (4).

For $p(x)=1$ we obtain the following result:

Theorem 2. *If $f\in C^n[a-h,\ a+h]$ and $r>1$, then*

$$\|f^{(n)}\|_r\geqq\frac{(n-1)!}{h^n}\left(\frac{nr-1}{r-1}\right)^{\frac{r-1}{r}}|M_nf(a)|.$$

The last Theorem (for $n=2$) represents the generalization of a result from [1].

From the Theorem 1 immediately follows:

Theorem 3. *Let functions f and p satisfy the conditions as in Theorem 1 and let $r>1$.*

1° *If $n=2k$ and $f^{(2i)}(a)=0$ $(i=1,\ldots,k-1)$, then*

$$\|f^{(2k)}\|_{r,p}\geqq\frac{(2k-1)!}{2h\,\sigma_{2k}(r)}|f(a+h)-2f(a)+f(a-h)|;$$

2° *If $n=2k+1$ and $f^{(2i-1)}(a)=0$ $(i=1,\ldots,k)$, then*

$$\|f^{(2k+1)}\|_{r,p}\geqq\frac{(2k)!}{2h\,\sigma_{2k+1}(r)}|f(a+h)-f(a-h)|.$$

3. Let $n \in \mathbf{N}$, $r > 1$ and $\gamma_n(r) = \left(\frac{1}{2h} \int_{a-h}^{a+h} \frac{|t-a|^{\frac{(n-1)r}{r-1}}}{p(t)^{\frac{1}{r-1}}} \, dt \right)^{\frac{r-1}{r}}$.

Theorem 4. *Let* $f \in C^n[a-h, a+h]$, $p \in C[a, b]$, $p(x) > 0$ $(x \in [a, b])$. *Then*

$$\|f^{(n)}\|_{r,p} \geqq \frac{(n-1)!}{2h\gamma_n(r)} \left| \sum_{k=0}^{n-1} \frac{h^k}{k!} \left(f^{(k)}(a-h) - (-1)^k f^{(k)}(a+h)\right) \right|, \tag{6}$$

with equality if and only if

$$f^{(n)}(t) = A \left(\frac{(t-a)^{n-1}}{p(t)} \right)^{\frac{1}{r-1}} \qquad (A \in \mathbf{R}) \tag{7}$$

when n is odd;

$$f^{(n)}(t) = A \left(\frac{|t-a|^{n-1}}{p(t)} \right)^{\frac{1}{r-1}} \operatorname{sgn}(t-a) \qquad (A \in \mathbf{R}), \tag{8}$$

when n is even.

Proof. Let $r > 1$ and $g(t) = p(t)^{\frac{1}{r}}$. According to HÖLLDER's inequality we have

$$\|f^{(n)}\|_{r,p} = \frac{1}{\gamma_n(r)} \left(\frac{1}{2h} \int_{a-h}^{a+h} (g(t)|f^{(n)}(t)|)^r \, dt \right)^{1/r} \left(\frac{1}{2h} \int_{a-h}^{a+h} \left(\frac{|t-a|^{n-1}}{g(t)} \right)^{\frac{r}{r-1}} dt \right)^{\frac{r-1}{r}}$$

$$\geqq \frac{1}{2h\gamma_n(r)} \int_{a-h}^{a+h} |t-a|^{n-1} |f^{(n)}(t)| \, dt,$$

where from it follows

$$\|f^{(n)}\|_{r,p} \geqq \frac{1}{2h\gamma_n(r)} \left| \int_{a-h}^{a+h} (t-a)^{n-1} f^{(n)}(t) \, dt \right|. \tag{9}$$

Since

$$\int_{a-h}^{a+h} (t-a)^{n-1} f^{(n)}(t) \, dt = (-1)^n (n-1)! \sum_{k=0}^{n-1} \frac{h^k}{k!} \left(f^{(k)}(a-h) - (-1)^k f^{(k)}(a+h)\right),$$

from (9) it follows (6).

Equality in (6) holds if and only if the function f satisfies the following conditions

$$|f^{(n)}(t)|^{r-1} = |A| \frac{|t-a|^{n-1}}{p(t)} \qquad (A \in \mathbf{R})$$

and

$$(t-a)^{n-1} f^{(n)}(t) \geqq 0 \qquad (\text{or} \leqq 0).$$

Hence, we obtain (7) and (8), which proves the Theorem.

From the Theorem 4 it follows:

Theorem 5. *Let* $f \in C^n[a-h, a+h]$ *and*

$$f^{(k)}(a-h) = (-1)^k f^{(k)}(a+h) \qquad (k = 1, \ldots, n-1),$$

then

$$\| f^{(n)} \|_r \geqq \frac{(n-1)!}{2h^n} \left(\frac{nr-1}{r-1} \right)^{\frac{r-1}{r}} |f(a+h) - f(a-h)|. \tag{10}$$

REMARK 1. For $n=2$ and $r=2k$, this result reduces to the Theorem 264 from [3].

REMARK 2. If $f \in C^1[a, b]$, according to Theorem 5 (for $n=1$), we have

$$\int_a^b |f'(x)|^r \, dx \geqq \frac{1}{(b-a)^{r-1}} |f(b) - f(a)|^r \qquad (r > 1).$$

This result is a particulary case of the inequality (4.12) from [4].

REFERENCES

1. R. Ž. ĐORĐEVIĆ and G. V. MILOVANOVIĆ: *On some generalizations of Zmorovič's inequality.* These Publications № **544** — № **576** (1976), 25—30.
2. V. A. ZMOROVIČ: *On some inequalities* (Russian). Izv. Polytehn. Inst. Kiev **19** (1956), 92—107.
3. G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA: *Inequalities.* Cambridge 1934.
4. G. V. MILOVANOVIĆ: *O nekim funkcionalnim nejednakostima.* These Publications № **599** (1977), 1—59.

Elektronski fakultet
18000 Niš, Jugoslavija

Građevinski fakultet
11000 Beograd, Jugoslavija