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639. ON SOME PROPERTIES OF g-CONVEX FUNCTIONS

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In the present paper we will show that some known properties of convex functions can be extended to the class of g-convex functions. An application of the results obtained in this paper is also given.

1. In this paper we will consider some properties of the class of g-convex functions. This class is defined in the following way.

Definition 1. *Suppose that the function* $g: I^2 \to \mathbf{R}$ *is given* ($I \subset \mathbf{R}$) *and that we have* $g(x, y) > 0$ *if* $x < y$ ($x, y \in I$). *For a function* f, *defined on* I, *we will say that it is g-convex on* I *if the inequality*

$$g(x_2, x_3) f(x_1) + g(x_3, x_1) f(x_2) + g(x_1, x_2) f(x_3) \geqq 0 \tag{1}$$

holds for an arbitrary points $x_1, x_2, x_3 \in I$ ($x_1 < x_2 < x_3$).

This definition was given, for the first time, by P. M. VASIĆ and J. D. KEČKIĆ in their paper [1]. In that paper they have considered some basic properties of g-convex functions as it is continuity, naturality of the definition 1 and they have also proved some inequalities for g-convex functions. The above definition 1, in a special cases, contains in it some of known generalizations of the class of convex functions in the usual sense.

As it is well known, for a real function f, we will say that it is convex on $I \subset \mathbf{R}$ if the inequality

$$f(tx + (1-t)y) \leqq tf(x) + (1-t) f(y)$$

holds true for an arbitrary pair of points $x, y \in I$ and every $t \in [0, 1]$. For a continuous and convex function f, defined on $\mathbf{R}$, the following theorem (see for example [2] pp. 286—287.) can easilly be proved:

Theorem 1. *If a function* f *is continuous and convex on* $\mathbf{R}$ *then there exist two real constants* A *and* B *such that we have*

$$f(x) \geqq Ax + B \tag{2}$$

for every $x \in \mathbf{R}$.

Naturaly, the following question can be proposed: is it possible to prove a theorem analogous to that above for g-convex functions?

In the present paper we will show that, under certain conditions for the function g, the above theorem 1 can be extended to the class of g-convex functions. Certainly, this generalization which will be given here, reduces to the above theorem 1 for suitable choice of g. In one of the following parts of this paper we will give also an application of this generalization of theorem 1.

2. As we have said above, we will consider only those functions $g\colon \mathbf{R}^2 \to \mathbf{R}$ for which the condition $g(x, y) > 0$ is valid if $x < y$. The following definition will be of use in the present paper.

Definition 2. *Suppose that the function g and the points $a, b \in \mathbf{R}$ are given. Every function L of the following form*

$$L(x) = C_1\, g(x, a) + C_2\, g(b, x) \tag{3}$$

where the constants $C_1, C_2 \in \mathbf{R}$ are arbitrary we will call — g-line on the points a and b.

Definition 3. *Let us suppose that the function g and the points $a, b \in \mathbf{R}$ are given. For a function g we will say that it is regular on the pair of points a and b if for every real constant m and every g-line L, on the points a and b $(a < b)$ there exists g-line L_0 on the points a and b such that we have*

$$m \geqq L(x) - L_0(x) \text{ for every } x \in [a, b], \tag{4}$$

and

$$L_0(x) \geqq 0 \text{ for every } x \notin [a, b]. \tag{5}$$

Examples of g-lines as examples of regular functions can easily be constructed. In the present paper, as it seems to us, the definitions of g-lines and regular functions, are given for the first time. On the other hand, in our paper [3], we have used these concepts implicitly. In the mentioned paper we have considered the epigraph of g-convex functions. The main result of this our paper can be expressed in the following form in terms of the concepts just defined.

Theorem 2. *Suppose that the function $g\colon I^2 \to \mathbf{R}$ satisfies the condition $g(x, y) > 0$ if $x < y$ $(x, y \in I)$. Let us suppose further that we have*

$$g(x, y) + g(y, x) = 0 \quad (x, y \in I). \tag{6}$$

Then a function f is g-convex on I if and only if this function satisfies the condition

$$f(x) \leqq L(x), \quad x \in (x_1, x_2) \tag{7}$$

for every pair of points $x_1, x_2 \in I$ $(x_1 < x_2)$ where g-line L on the points x_1 and x_2 is given by

$$L(x) = \frac{f(x_1)}{g(x_1, x_2)}\, g(x, x_2) + \frac{f(x_2)}{g(x_1, x_2)}\, g(x_1, x). \tag{8}$$

Theorem 3. *Let us suppose that the function $g\colon I^2 \to \mathbf{R}$ satisfies the conditions of theorem 2. A function $f\colon I \to \mathbf{R}$ is g-convex on I if and only if the epigraph of f satisfies the condition: for every pair of points (x_1, y_1), (x_2, y_2) $(x_1 < x_2)$ from the epigraph of the function f, every point of the form $(x, L(x))$, where we have $x_1 \leqq x \leqq x_2$, also belongs to the epigraph of the function f, where L is g-line on the points x_1 and x_2 having the following form*

$$L(x) = \frac{y_1}{g(x_1, x_2)}\, g(x, x_2) + \frac{y_2}{g(x_1, x_2)}\, g(x_1, x). \tag{9}$$

Let us return now to the theorem 1. We will prove the following theorem, which is analogous to that of 1, and which will contain some properties of g-convex functions.

Theorem 4. *Suppose that the function* g: $\mathbf{R}^2 \to \mathbf{R}$ *satisfies the condition* $g(x, y) > 0$ *for* $x < y$ $(x, y \in \mathbf{R})$. *Suppose further that the function* f *is* g*-convex and continuous on* $\mathbf{R}$. *Then, if there exist a pair of points* $a, b \in \mathbf{R}$ *on which the function* g *is regular function, there exist also a* g*-line* $x \mapsto L(x)$ *such that we have*

$$f(x) \geqq L(x) \tag{10}$$

for all $x \in \mathbf{R}$.

Proof. We can suppose, without lose of generality, that $a<b$. Since, f is g-convex function on $\mathbf{R}$, from (1) and $x_1=a$, $x_2=b$ and $x_3=x>b$ it follows that the inequality

$$f(x) \geqq L_1(x) \tag{11}$$

holds true for all $x>b$, where L_1 is *g-line* defined by

$$L_1(x) = -\frac{f(a)}{g(a, b)} g(b, x) - \frac{f(b)}{g(a, b)} g(x, a). \tag{12}$$

Similarly, since f is g-convex function on $\mathbf{R}$, if we take $x_1=x<a$, $x_2=a$, $x_3=b$ on the basis of (1) we obtain that the inequality (11) is valid for every $x<a$, where the g-line L_1 is given by (12). In such a way we conclude that the inequality (11) holds for every $x \notin [a, b]$.

In virtue of the suppositions of theorem 4, g-convex function f is continuous on $\mathbf{R}$. That means that there exists $m \in \mathbf{R}$ such that

$$m = \min_{a \leq x \leqq b} f(x). \tag{13}$$

By using the supposition that g is regular function on the pair of points a and b, from the above definition 3 it follows that there are two constants C_1 and C_2 such that we have

$$m \geqq L_1(x) - L_0(x), \quad x \in [a, b] \tag{14}$$

and that

$$L_0(x) \geqq 0, \quad x \notin [a, b] \tag{15}$$

where we have taken

$$L_0(x) = C_1 g(b, x) + C_2 g(x, a). \tag{16}$$

We shall further consider the g-line L, defined by

$$L(x) = L_1(x) - L_0(x) = (D_1 - C_1) g(b, x) + (D_2 - C_2) g(x, a) \tag{17}$$

with $D_1 = \frac{-f(a)}{g(a, b)}$ and $D_2 = \frac{-f(b)}{g(a, b)}$. We shall prove that the g-convex function f and the g-line L, defined by (17) satisfy the inequility (10) for all $x \in \mathbf{R}$.

On the basis of (13), (14) and (17) we have that the inequality (10) is valid for all $x \in [a, b]$ since

$$f(x) \geqq m \geqq L_1(x) - L_0(x) = L(x)$$

holds for all $x\,(a\leqq x\leqq b)$. On the other side if we have $x\notin[a, b]$ then in virtue of (11) and (17) it follows that

$$f(x)\geqq L_1(x)\geqq L_1(x)-L_0(x)=L(x)$$

because of $L_0(x)\geqq 0$ (see (15)) for $x\notin[a, b]$. This completes the proof of theorem 4.

In the above theorem we have supposed that the function f is continuous on $\mathbf{R}$. In paper [1] P. M. Vasić and J. D. Kečkić gave a sufficient condition for the continuity of g-convex function. Namely, g-convex function is continuous if g is continuous function and if the same function satisfies the condition (6) (see theorem 2 from [1]).

3. In this part of the present paper we will give an application of theorem 4.

Namely, from theorem 1 directly follows the following theorem (see [2] page 278).

Theorem 5. *Let us suppose that a function f satisfies the conditions of theorem* 1. *If the functions $t\mapsto x(t)$ and $t\mapsto p(t)$ are defined on $[a, b]$, $p(t)\geqq 0$ for $a\leqq t\leqq b$, x is measurable and finite almost everywhere and the functions p and px are integrable on $[a, b]$ then the integral*

$$\int_a^b p(t)f(x(t))\,dt \tag{18}$$

exists and its value is finite or equals $+\infty$.

In the present paper we will give the following generalization of the preceding theorem.

Theorem 6. *Suppose that the function $g\colon \mathbf{R}^2\to\mathbf{R}$ satisfies the condition $g(x, y)>0$ if we have $x<y$ and that a continuous function f is g-convex on $\mathbf{R}$. Let also $t\mapsto p(t)$ and $t\mapsto x(t)$ be a functions defined on $[a, b]$, where x is measurable and finite almost everywhere on the same segment. If further there exist two points $t_1, t_2\in[a, b]$ such that we have $x(t_1)\neq x(t_2)$ and such that g is regular function over the points $x(t_1)$ and $x(t_2)$, and if the functions $t\mapsto p(t)\,g(x(t), x(t_1))$ and $t\mapsto p(t)\,g(x(t_2), x(t))$ are summable on $[a, b]$ then the integral* (18) *has its numerical value which is finite or equals $+\infty$.*

Proof. Without lose of generality we can suppose that the points t_1 and t_2 are chosen in such a way that we have $x(t_1)<x(t_2)$, because of we have supposed that $x(t_1)\neq x(t_2)$. The function $t\mapsto y(t)=f(x(t))$ is finite almost everywhere on the segment $[a, b]$ and it is measurable on the same segment. So, by Frechet's theorem there exists a sequence of continuous functions which converges almost everywhere in $[a, b]$ to the function x. This sequence will be denoted by $x_n(t)$, $t\in[a, b]$. Therefrom it follows that the sequence $f(x_n(t))$ of continuous functions converges to the function $y(t)$. Let us denote

$$y_+(t)=\begin{cases} y(t), & y(t)\geqq 0\\ 0, & y(t)<0\end{cases},\qquad y_-(t)=\begin{cases} 0, & y(t)\geqq 0\\ -y(t), & y(t)<0\end{cases},$$

wherefrom we have $y(t)=y_+(t)-y_-(t)$ for all $t\in[a, b]$. Let us put $A=\{t\in[a, b]\mid y(t)<0\}$. On the basis of the suppositions of theorem 6 and on the basis of theorem 4, it follows that there exist two constants C_1 and C_2 such that we have

$$y(t)=f(x(t))\geqq L(x(t)) \tag{19}$$

for every $t \in [a, b]$, where L is defined by $L(x) = C_1 g(x, x(t_1)) + C_2 g(x(t_2), x)$. The integral (18) has a finite value if and only if the difference

$$\int_a^b p(t) y_+(t)\,\mathrm{d}t - \int_a^b p(t) y_-(t)\,\mathrm{d}t \tag{20}$$

has a finite value. In virtue of (19) it follows that

$$0 \leqq y_-(t) \leqq -L(x(t)) \tag{21}$$

for every $t \in A$. Since we have $p(t) \geqq 0$, $t \in [a, b]$ we find that

$$\int_A p(t) y_-(t)\,\mathrm{d}t \leqq -\int_A p(t) L(x(t))\,\mathrm{d}t < +\infty \tag{22}$$

because we have supposed that the functions

$$t \mapsto p(t) g(x(t), x(t_1)) \text{ and } t \mapsto p(t) g(x(t_2), x(t))$$

are summable on $[a, b]$ which implies their summability on the set A. From (22) it follows that

$$\int_a^b p(t) y_-(t)\,\mathrm{d}t < +\infty \tag{23}$$

because of $p(t) y_-(t) = 0$ if we have $t \notin A$. From (23) follows that the difference (20) is finite or equal to $+\infty$, which completes the proof.

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