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638. ON THE EPIGRAPH OF g-CONVEX FUNCTIONS

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In paper [1] P. M. VASIĆ and J. D. KEČKIĆ have defined a class of real functions which, in a special case, contains the class of convex functions and at the same time also contains some of the known generalizations of this class. This definition will be given here.

Definition 1. *Let $g: I^2 \to I$ be a given function such that $g(x, y) > 0$ for $x < y$ $(x, y \in I \subset \mathbf{R})$. For a function f we say that it is convex with respect to g on I (g-convex on I) if the condition*

$$g(x_2, x_3) f(x_1) + g(x_3, x_1) f(x_2) + g(x_1, x_2) f(x_3) \geqq 0 \tag{1}$$

is valid for arbitrary points $x_1, x_2, x_3 \in I$ $(x_1 < x_2 < x_3)$.

In the mentioned paper [1] certain properties of g-convex functions were examined. Namely, it was examined: (i) for which g the definition 1 is natural; (ii) continuity of g-convex functions; (iii) an inequality was proved, which in special case reduces to the known inequality of M. PETROVIĆ. Of course, it is necessary to introduce additional conditions on the function g, as it was done in the paper [1]. Let us remark that the assumption that the antidomen of g must be also the set $I \subset R$ is not necessary, as it was assumed in definition 1 and theorems from paper [1]. It is sufficient to assume that $g(x, y) \in R$ for $x, y \in I$.

In the case when the function g is chosen in such a way that $g(x, y) = y - x$, aforementioned definition 1, i. e. inequality (1), reduces to

$$f(tx + (1-t)y) \leqq t f(x) + (1-t) f(y) \tag{2}$$

where $x, y \in I$ and $t \in [0, 1]$; hence g-convex functions reduce to ordinary convex functions.

In this article the following, well known definition of epigraph of a real function will also be of use.

Definition 2. *Suppose that a real function f is defined on $I \subset \mathbf{R}$. The epigraph of f (denoted by* epi (f)*) is a subset of $\mathbf{R}^2$ defined by*

$$\operatorname{epi}(f) = \{(x, y) \mid f(x) \leqq y,\ x \in I\}. \tag{3}$$

For a set $S \subset \mathbf{R}^2$, we will say that it is convex if from $X, Y \in S$ follows $tX + (1-t) Y \in S$ for every $t \in [0, 1]$ (addition of points from $\mathbf{R}^2$ and multiplication by a scalar of points from $\mathbf{R}^2$ is defined in the usual way).

The following theorem which gives a characterization of the epigraph of a convex function (i. e. a function which satisfies the inequality (2)) is well-known.

Theorem 1. *A real function is convex on I if and only if its epigraph is a convex set.*

In this paper we shall prove a theorem which, analogously to the above theorem 1, characterizes the epigraph of a g-convex function. At the same time, we will assume that the function g, besides the conditions of the definition 1, also satisfies the condition

$$g(x, y)+g(y, x)=0 \tag{4}$$

for every pair of points $x, y \in I$ (as it was assumed in theorem 2 from paper [1]).

Naturally, we also have to generalize the notion of convex set.

Definition 3. *Suppose that the function $g: I^2 \to \mathbf{R}$ satisfies the condition* (4) *and that $g(x, y)>0$ if $x<y$ $(x, y \in I)$. For a set $S \subset \mathbf{R}^2$ we will say that it is g-convex if from the assumption (x_1, y_1), $(x_2, y_2) \in S$ $(x_1<x_2)$ it follows that*

$$\left(x, \frac{g(x, x_2)}{g(x_1, x_2)} y_1+\frac{g(x_1, x)}{g(x_1, x_2)} y_2\right) \in S \tag{5}$$

for every x $(x_1<x<x_2)$.

We do not pretend that this definition is formulated for the first time in this article, having in mind the large number of generalizations of the concept of a convex set, which can be found in mathematical literature. For our purpose it is essential only to underline here that this definition in a special case when $g(x, y)=y-x$ contains the classical concept of a convex set, and on the other hand this definition makes it possible to prove the following theorem.

Theorem 2. *Suppose that the function $g: I^2 \to \mathbf{R}$ satisfies the condition* (4) *and that $g(x, y)>0$ if $x<y$ $(x, y \in I)$. Then the function $f: I \to \mathbf{R}$ is g-convex if and only if* epi (f) *is a g-convex set.*

Proof. (i) Suppose that epi (f) is a g-convex set, and let us prove that the function f is g-convex. We will consider the points $x_1, x_2 \in I$ $(x_1<x_2)$. On the basis of definition 3 we have that $(x_1, f(x_1))$, $(x_2, f(x_2)) \in$ epi (f). Since epi (f) is a g-convex set, it follows on the basis of definition 3, that (5) holds for every x $(x_1<x<x_2)$. This means, in virtue of definition 2, that inequality

$$f(x) \leqq \frac{g(x, x_2)}{g(x_1, x_2)} f(x_1)+\frac{g(x_1, x)}{g(x_1, x_2)} f(x_2) \tag{6}$$

is valid for all x $(x_1<x<x_2)$. Since $g(x_1, x_2)>0$ which is implied by $x_1<x_2$, and since $g(x_1, x_2)=-g(x_2, x_1)$ which is implied by (4), it follows, from the above inequality (6), that

$$g(x, x_2) f(x_1)+g(x_2, x_1) f(x)+g(x_1, x) f(x_2) \geqq 0 \tag{7}$$

whenever $x_1<x<x_2$. In this way, using (1) we can conclude that the function f is g-convex.

(ii) Suppose, furter, that the function f is g-convex. We will prove now that the set epi (f) is also g-convex. We assume that (x_1, y_1), $(x_2, y_2) \in$ epi (f) sand that $x_1<x<x_2$. Then, on the basis of the definition 1 it follows that (7) holds, where-

from in a similar way as before, we get (6). Since, in virtue of definition 2 we have $f(x_1) \leqq y_1$, $f(x_2) \leqq y_2$ and since $\frac{g(x, x_2)}{g(x_1, x_2)} > 0$, $\frac{g(x_1, x)}{g(x_1, x_2)} > 0$ from (6) follows that

$$f(x) \leqq \frac{g(x, x_2)}{g(x_1, x_2)} y_1 + \frac{g(x_1, x)}{g(x_1, x_2)} y_2$$

holds for every x $(x_1 < x < x_2)$. In other words, the point (5) belongs to the set epi (f), which completes the proof.

The geometrical meanning of the definition 3 and at the same time of the theorem 2 is easily obtained. Namely, let $S \subset \mathbf{R}^2$ be a set which is g-convex and let the points (x_1, y_1), $(x_2, y_2) \in S$ be given and fixed, where we suppose that $x_1 < x_2$. We consider further the arc of the curve, whose equation is given by

$$F(x) = \frac{g(x, x_2)}{g(x_1, x_2)} y_1 + \frac{g(x_1, x)}{g(x_1, x_2)} y_2 \tag{8}$$

for all $x \in (x_1, x_2)$. We conclude that g-convexity of the set S is equaivalent to the condition that for every pair of points (x_1, y_1), $(x_2, y_2) \in S$ all the points of the curve (8) belong to the same set S for all $x \in (x_1, x_2)$ (i. e. all the points of the mentioned curve between the points (x_1, y_1) and (x_2, y_2) belong to S).

In particular cases (by appropriate choices of g) our theorem 2 caracterizes epigraphs of functions convex in the sense of G. Valiron **[2]**, **[3]**, E. Phragmen and E. Lindelöf **[4]** and И. Е. Овчаренко **[5]**.

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