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636. CUBIC SPACE CURVES ON CAYLEY'S CYLINDROID

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1. The cylindroid C of CAYLEY (also called PLÜCKER's conoid) is a much studied metrically special ruled cubic surface which is an important subject in BALL's theory of screws. On a suitably chosen Cartesian frame the equation of C is, for homogeneous coordinates,

$$(x^2+y^2)\,z=2\,dxyw, \qquad d>0. \tag{1.1}$$

C is invariant for the reflection into OZ. In a horizontal plane $z=mw$ we have two generators intersecting at a point of OZ. In the central plane $z=0$ they are the X- and Y-axis. For $-d<m<d$ the two generators are real and distinct. For $d=\pm m$ they coincide with the dorsal lines of C, the dorsal planes being $z=\pm dw$. All generators intersect the line l with the equations $z=w=0$. The intersection of C and the plane W at infinity consists of l and the two conjugate imaginary lines s_1 and s_2, given by $x^2+y^2=w=0$, intersecting at the point at infinity of OZ.

On C there are ∞^1 straight lines and ∞^2 ellipses (the intersections of C and arbitrary planes through a generator) and ∞^3 plane cubics. We ask whether there exist on C real non-degenerate twisted cubics.

2. Such a cubic k is represented if x, y, z, w are given as four linearly independent cubic functions of a parameter t. Obviously of the intersections of k and W one, denoted by A, is on l and two, B_1 and B_2, on s_1 and s_2, respectively. The parameter t may be chosen such that A corresponds to $t=0$ and the B's to $t=\pm i$. Hence we can take $2\,dw=t\,(1+t^2)$; z must have the factor t. We take $z=t^2$, a choice justified by the result. The condition that k is on C reads $(x^2+y^2)\,t^2=xyt\,(1+t^2)$ which is satisfied by $x=1,\ y=t$ and by $x=t$, $y=1$. By this trial and error method we have found two cubics on C:

$$k_1:\quad x=1,\ y=t,\ z=t^2,\ 2\,dw=t\,(1+t^2), \tag{2.1}$$

$$k_2:\quad x=t,\ y=1,\ z=t^2,\ 2\,dw=t\,(1+t^2). \tag{2.2}$$

It is easy to verify that in both cases x, y, z, w are linearly independent, which implies that we deal with space curves.

3. k_1 and k_2 are special cubics on C, both with simple equations. The transformation $t'=-t$ shows that k_1 is invariant for the reflection into the Y-axis. It passes (for $t=0$) through $A=(1, 0, 0, 0)$, for $t=\pm i$ through $(1, \pm i, -1, 0)$ and for $t=\infty$ through the origin O. A horizontal plane $V: z=mw$ intersects k_1 at A and at two more, finite, points E_1 and E_2, determined by $mt^2-2\,dt+m=0$, one on each of the two generators in V. For $m=0$ one coincides with O; for $m=d$ the generators coincide with the dorsal line $x-y=z-dw=0$ and E_1, E_2 coincide at $(1, 1, 1, d^{-1})$, for $m=-d$ they coincide at $(-1, 1, -1, d^{-1})$ on the lower dorsal line $x+y=z+dw=0$. For k_2 we have analogous results; k_2 follows from k_1 by interchanging x and y.

4. It is well-known that ∞^2 quadrics pass through a twisted cubic. From (2. 1) it follows

$$1:t:t^2:t^3=x:y:z:(2dw-y). \tag{4.1}$$

This implies that through k_1 pass all quadrics of the linear system

$$\alpha Q_1+\beta Q_2+\gamma Q_3=0, \tag{4.2}$$

with

$$Q_1=xz-y^2,\ Q_2=y(2\,dw-y)-z^2,\ Q_3=x(2\,dw-y)-yz, \tag{4.3}$$

$Q_i=0$ $(i=1, 2, 3)$ being linearly independent quadrics. The intersection of C and a quadric Q of the set (4.2) is a curve of degree six. As k_1 is on Q and on C it belongs to the intersection and the remaining part is also of the third order. It cannot be a plan cubic because it lies on a quadric. Furthermore it is easy to verify that k_2, given by (2.2), does not lie on any quadric (4.2). *Hence there exist two systems, both of ∞^2 cubic space curves on the cylindroid.*

5. It seems to ask for much algebra to determine explicitly the cubic curves in terms of α, β, γ. Therefore we restrict ourselves to a subset of (4.2), the cones of the set. Let T_0 (with $t=t_0$) be a fixed point and T a variable point on k_1. We consider the cone through k_1 with vertex T_0. A generator of the cone is represented by

$$x=1+u,\ y=t+ut_0,\ z=t^2+ut_0^2,\ 2\,dw=t(1+t^2)+ut_0(1+t_0^2), \tag{5.1}$$

with t fixed and u variable. This line has three intersections with C, two of them being T_0 and T, corresponding to $u=\infty$ and $u=0$ respectively; let T' be the third intersection. If we substitute (5.1) into (1.1), the coefficient of u^3 and the constant term vanish and third root is

$$u=(t^2+t_0t-1)/(-t_0t-t_0^2+1). \tag{5.2}$$

If we substitute this in (5.1), we obtain after some algebra the locus of T' for variable t:

$$x=t+t_0,\ y=1,\ z=(t+t_0)(-t_0t+1),\ 2\,dw=(-t_0t+1)\left((t+t_0)^2+1\right), \tag{5.3}$$

which for any value of t_0 represents a twisted cubic on C; *we have derived therefore explicitly a system of ∞^1 such curves.*

A curve (5.3) could be considered as the projection of k_1 from its point T_0 as the centre, on C itself. It intersects the line l (for $t=t_0^{-1}$) at $A=(t_0^2+1, t_0, 0, 0)$; for the two other intersections with W we have $x^2+y^2=0$ as it should be. The intersections of (5.3) and the dorsal plane $z=dw$ are the point A and the point (corresponding to $t=1-t_0$) counted twice, and analogously for the lower dorsal plane.

The system of curves derived from k_2 are found in a similar way.

6. An alternative approach to our problem would be the following. A representation of the surface C by means of two parameters λ and μ is, for instance,

$$x=\lambda(\lambda^2+\mu^2),\ y=\mu(\lambda^2+\mu^2),\ z=2d\lambda\mu,\ w=(\lambda^2+\mu^2). \tag{6.1}$$

A curve on C is defined if λ and μ are given functions of t. We obtain a cubic if λ and μ are linear functions. But if, for instance, $\lambda=pt+q$, $\mu=rt+s$, the relation

$$rx-py+(ps-qr)w=0 \tag{6.2}$$

holds, which implies that we deal with a planar cubic. Our curve k_1 may be found from (6.1) but in a more complicated way. Indeed we obtain (2.1) if

$$\lambda=t^{-1}(1+t^2)^{-1},\ \mu=(1+t^2)^{-1}.$$

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