UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 634 — № 677 (1979), 25—32.

635. SOME EXTREMAL PROPERTIES OF ORTHOGONAL POLYNOMIALS

Branko D. Rakovich and *Petar M. Vasić*

ABSTRACT. Using a particular way of normalizing the orthogonal polynomials, which is most commonly encountered in the synthesis of filtering networks in communication and electronic engineering, two theorems concerning the extremal properties of orthogonal polynomials are first proved. The results are then applied to find the minimum value and the minimizing function for various definite integrals involving weight functions of classical orthogonal polynomials.

1. Introduction. Orthogonal polynomials are of considerable importance in many branches of science and engineering since they represent an indispensable analytical tool for solving various approximation problems. In particular, the approximation problem in the synthesis of electric filters, that the essential parts of many systems in electronics and communication engineering, consists of finding a physical realizable rational function of frequency that shall meet a prescribed set of specifications with regard to its amplitude and/or phase characteristics. Also, if the network is synthesized for use in pulse transmission systems additional contraints may be imposed on the shape of the time domain responses of the network since, for example, overshoot is undesirable and must be kept to within a prescribed value. Although the criterion for best approximation largely depends on the intended application the lest-mean square error norm is often employed and this accounts for widespread use of all type of classical and some other classes of orthogonal polynomials in filter synthesis.

In many instances the filter function takes the form of a reciprocal of a polynomial the amplitude of which is required to approximate zero in the useful frequency interval and to deviate as much as possible from zero in the rest of the frequency band. If the criterion for best approximation is stated in terms of a least-mean square norm the minimum of the amplitude squared function integrated over the useful band in association with a suitable chosen weight function leads to the minimization of the insertion power loss in the useful band. This, from the physical point of view, represents a well defined design objective. In low-pass filters the useful band is defined as the frequency interval between zero and the frequency at which the characteristic function of the filter reaches the value of 1. Thus, in order to find the minimizing function and the minimum value of the error integral the filter function is first expanded into a series of orthogonal polynomials, and, by a simple frequency transformation, the useful band is made to coincide with the orthogonality interval of the orthogonal polynomials. Then the minimization of the error integral can be performed by standard method using the orthogonality relations for orthogonal polynomials. This process is time consuming and cannot be solved in each specific case, even for filtering functions of lower order, without resorting to tedious numerical computation using a digital computer. However, the closed form solution of this problem can be made possible by the use of some extremal properties of orthogonal polynomials that will be proved in the following sections. In addition, the application of the main result will be shown to include as special cases the minimum values of some definite integrals recently obtained by MORDELL [2], [3].

2. Preliminaires. Let $x \mapsto w(x)$ be a nonnegative function on the interval $[a, b]$ such that

$$\int_a^b w(x)\, x^r \,\mathrm{d}x$$

exists for $r \geqq 0$ and consider the definite integral of the form

$$I_n = \int_a^b w(x)\,(b_0 + b_1 x + \cdots + b_n x^n)^2 \,\mathrm{d}x. \tag{2.1}$$

The problem to be solved is to determine the polynomial $x \mapsto f_n(x)$ of order n which minimizes the integral (2.1) under the constraint that $f_n(p) = 1$ for any given real number p. Since the integrand is nonnegative for any value of $x \in [a, b]$ such a minimum value does exist.

If $Q_0, Q_1, Q_2, \ldots$ is a set of orthogonal polynomials associated with the weight function $x \mapsto w(x)$ on $[a, b]$, the polynomial f_n can be expanded into a finite series of $x \mapsto Q_i(x)$ so that

$$I_n = \int_a^b w(x) \Big(\sum_{i=0}^{n} a_i Q_i(x) \Big)^2 \mathrm{d}x. \tag{2.2}$$

Using standard minimization technique [5] and starting from

$$\varphi(a_0, a_1, \ldots, a_n, \beta) = \int_a^b w(x) \Big(\sum_{i=0}^{n} a_i Q_i(x) \Big)^2 \mathrm{d}x + \beta \Big(\sum_{i=0}^{n} a_i Q_i(p) - 1 \Big) \tag{2.3}$$

where β is the Lagrangian multiplier, we have

$$\begin{aligned} \frac{\partial \varphi}{\partial a_i} &\equiv 2 \int_a^b w(x)\, a_i\, Q_i(x)^2 \,\mathrm{d}x + \beta\, Q_i(p) = 0, \\ &\sum_{i=0}^{n} a_i Q_i(p) = 1. \end{aligned} \tag{2.4}$$

Denoting by

$$h_i = \int_a^b w(x)\, Q_i(x)^2 \,\mathrm{d}x \tag{2.5}$$

we easily find

$$a_i = \frac{Q_i(p)}{h_i} \, \frac{1}{\displaystyle\sum_{j=0}^{n} \frac{Q_j(p)^2}{h_j}}, \tag{2.6}$$

so that the minimum value M of the integral (2.1) under the aforementioned constraint is

$$M=\frac{1}{\sum_{i=0}^{n}\frac{Q_i(p)^2}{h_i}}. \tag{2.7}$$

3. Main results.

Theorem 1. *If c is finite and $c\leqq a$ or $c\geqq b$, the integral*

$$I_n=\int_a^b w(x)f_n(x)^2\,dx, \tag{3.1}$$

where f_n is any real polynomial of degree n such that $f_n(c)=1$, reaches its minimum value if and only if $f_0, f_1, f_2, \ldots$ form a set of orthogonal polynomial on $[a, b]$ with respect to the weight function $x\mapsto(x-c)w(x)$ for $c\leqq a$, and with respect to the weight function $x\mapsto(c-x)w(x)$ for $c\geqq b$.

Proof. Suppose $c(\leqq a)$ is finite. From (2.2) and (2.6) the polynomial f_n that minimizes the definite integral (2.1), subject to the condition $f_n(c)=1$, is

$$f_n(x)=\left(\sum_{j=0}^{n}\frac{Q_j(c)^2}{h_j}\right)^{-1}\sum_{i=0}^{n}\frac{Q_i(c)^2}{h_i}Q_i(x). \tag{3.2}$$

But, from the CHRISTOFFEL-DARBOUX theorem [4, p. 42], it follows that

$$\sum_{i=0}^{n}\frac{Q_i(c)Q_i(x)}{h_i}=\frac{k_n}{k_{n+1}h_n}\,\frac{Q_n(c)Q_{n+1}(x)-Q_{n+1}(c)Q_n(x)}{x-c}, \tag{3.3}$$

where k_n is the coefficient of x^n in $x\mapsto Q_n(x)$, so that the minimizing polynomial takes the form

$$f_n(x)=K\frac{Q_n(c)Q_{n+1}(x)-Q_{n+1}(c)Q_n(x)}{x-c}\quad(K=\text{const}). \tag{3.4}$$

Now we employ the CHRISTOFFEL formula [4, p. 28] stating that if $Q_0, Q_1, Q_2, \ldots$ form a set of orthogonal polynomials associated with the weight function w on $[a, b]$, then the polynomials $R_0, R_1, R_2, \ldots,$ where

$$R_n(x)=K\frac{Q_n(c)Q_{n+1}(x)-Q_{n+1}(c)Q_n(x)}{x-c}\qquad(n=0, 1, 2, \ldots) \tag{3.5}$$

are orthogonal on the same segment $[a, b]$ in respect of the weight function $x\mapsto(x-c)w(x)$. This evidently completes the proof of Theorem 1 for $c\leqq a$. A similar result holds if $c(\geqq b)$ is finite.

An immediate consequence of Theorem 1 is the following, more general, result:

Theorem 2. *Let g be an increasing function on $[a, b]$ and w a nonnegative weight function on the same interval such that the integral*

$$\int_a^b w(x)g(x)^r\,dx\qquad(r\geqq 0)$$

exists. Then the sequence of functions $x \mapsto f_0(g(x))$, $x \mapsto f_1(g(x))$, $x \mapsto f_2(g(x))$, … *that minimizes the integrals*

$$I_n = \int_a^b w(x) f_n(g(x))^2 \,\mathrm{d}x \qquad (n = 0, 1, 2 \ldots) \tag{3.6}$$

where f_n *is a real polynomial of degree* n, *forms an orthogonal system on* $[a, c]$ *associated with the weight function* $x \mapsto (g(x) - c)\, w(x)$ *for* $c \leqq g(a)$, *or associated with* $x \mapsto (c - g(x))\, w(x)$ *for* $c \geqq g(b)$.

Proof. Substituting $g(x) = t$ in (3.6) we have

$$I_n = \int_{g(a)}^{g(b)} p(t) f_n(t)^2 \,\mathrm{d}t \qquad (n = 0, 1, 2, \ldots), \tag{3.7}$$

where

$$p(t) = \frac{w(g^{-1}(t))}{g'(g^{-1}(t))}. \tag{3.8}$$

According to Theorem 1, the functions that minimize (3.6) form an orthogonal system on $[g(a), g(b)]$ in respect of $t \mapsto (t - c)\, p(t)$ for $c \leqq g(a)$, i.e.,

$$\int_{g(a)}^{g(b)} (t - c)\, p(t) f_j(t) f_k(t) \,\mathrm{d}t = 0 \qquad (j, k = 0, 1, 2, \ldots; j \neq k). \tag{3.9}$$

Now comming back to the old variable x, $t = g(x)$, we get

$$\int_a^b (g(x) - c)\, w(x) f_j(g(x)) f_k(g(x)) \,\mathrm{d}x = 0 \qquad (j, k = 0, 1, 2, \ldots; j \neq k), \tag{3.10}$$

and this completes the proof of Theorem 2.

4. Applications. If the polynomials Q_i $(i = 0, 1, 2, \ldots)$ are orthogonal on $[a, b]$ in respect of w, then from Theorem 1, we have for $c \geqq b$,

$$\min I_n = \int_a^b w(x) \left(\sum_{i=0}^{n} \frac{Q_i(c) M}{h_i} Q_i(x) \right)^2 \mathrm{d}x \tag{4.1}$$

$$= \int_a^b w(x) \left(\frac{R_n(x)}{R_n(c)} \right)^2 \mathrm{d}x,$$

where $R_0, R_1, R_2, \ldots$ are the polynomials orthogonal with respect to $x \mapsto (c - x) w(x)$ on $[a, b]$. Since for any given nonnegative weight function w and any prescribed segment $[a, b]$ there is only one set of orthogonal polynomials which are determined to within a constant multiplier, it follows that

$$\sum_{i=0}^{n} \frac{Q_i(c) M}{h_i} Q_i(x) = \frac{R_n(x)}{R_n(c)}, \tag{4.2}$$

and by comparing the coefficients of x^n, we find

$$M=\frac{h_n k_n^R}{Q_n(c) R_n(c) k_n^Q}, \tag{4.3}$$

where k_n^Q and k_n^R are the coefficients of x^n in Q_n and R_n respectively. Hence, from (4.2) and (4.3)

$$R_n(x)=\sum_{i=0}^{n} \frac{Q_i(c)}{Q_n(c)} \frac{h_n}{h_i} \frac{k_n^R}{k_n^Q} Q_i(x). \tag{4.4}$$

Special cases. 1° Suppose $f_n(x)=a_0+a_1 x+\cdots+a_n x^n$ is a real nth order polynomial such that $f_n(1)=1$, $w(x)=(1-x)^\alpha(1+x)^\beta$ and $a=-1$, $b=1$, then substituting in (4.1)

$$\begin{aligned} &Q_n(1)=P_n^{\alpha,\beta}(1)=\binom{n+\alpha}{n}, \quad R_n(1)=P_n^{\alpha+1,\beta}(1)=\binom{n+\alpha+1}{n}, \\ &k_n^Q=2^{-n}\binom{2n+\alpha+\beta}{n}, \quad k_n^R=2^{-n}\binom{2n+\alpha+\beta+1}{n}, \end{aligned} \tag{4.5}$$

whhile α, $\beta>-1$, and

$$h_n=\frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}, \tag{4.6}$$

while $P_n^{\alpha,\beta}$ is the JACOBI polynomial, we obtain

$$\begin{aligned} &\min \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta (a_0+a_1 x+\cdots+a_n x^n)^2\, dx \\ &=\frac{1}{\binom{n+\alpha+1}{n}^2} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{\alpha+1,\beta}(x)^2\, dx \\ &=\frac{2^{\alpha+\beta+1}\, n!\, \Gamma(\alpha+1)\, \Gamma(\alpha+2)\, \Gamma(n+\beta+1)}{\Gamma(n+\alpha+2)\, \Gamma(n+\alpha+\beta+2)}. \end{aligned} \tag{4.7}$$

Also, from (4.4) the following important identity for the JACOBI polynomials [4] is recovered

$$\sum_{i=0}^{n} \frac{(2i+\alpha+\beta+1)\,\Gamma(i+\alpha+\beta+1)}{\Gamma(i+\beta+1)} P_i^{\alpha,\beta}(x)=\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)} P_n^{\alpha+1,\beta}(x). \tag{4.8}$$

From (4.7) some known results regarding the minimum values of some classes of definite integrals immediately follows. Thus, for example, if $\alpha=\beta=0$, we have

$$\int_{-1}^{1} (a_0+a_1 x+\cdots+a_n x^n)^2\, dx \geq \frac{2}{(n+1)^2}. \tag{4.9}$$

The last result is obtained for $f_n(1)=1$ but it still holds good if the polynomial f_n reaches the value of 1 at any point in the interval $[-1, 1]$. To prove this let $f_n(c)=1$ $(-1\leqq c\leqq 1)$ and substituting $2x=(c-1)t+(c+1)$, we have

$$\text{(4.10)}\qquad \int_c^1 f_n(x)^2\,dx=\frac{1-c}{2}\int_{-1}^1 f_n\left(\frac{c-1}{2}t+\frac{c+1}{2}\right)^2 dt.$$

If f_n is a polynomial of degree n, so is the polynomial

$$g_n(x)=f_n\left(\frac{c-1}{2}x+\frac{c+1}{2}\right).$$

Since $f_n(c)=g_n(1)=1$, we have from (4.9) and (4.10)

$$\text{(4.11)}\qquad \int_c^1 (a_0+a_1x+\cdots+a_nx^n)^2\,dx\geqq\frac{1-c}{(n+1)^2}.$$

This result is due to F. Bowman for $c=0$ (see Mordell [2]). In a similar way we find

$$\text{(4.12)}\qquad \int_{-1}^c (a_0+a_1x+\cdots+a_nx^n)^2\,dx\geqq\frac{1+c}{(n+1)^2},$$

so that from (4.11) and (4.12) it follows imediately that if a real polynomial $f_n(x)=a_0+a_1x+\cdots+a_nx^n$ reaches the value of 1 anywhere on the segment $[-1, 1]$, then

$$\text{(4.13)}\qquad \int_{-1}^1 (a_0+a_1x+\cdots+a_nx^n)^2\,dx\geqq\frac{2}{(n+1)^2}.$$

Except for an obvious error, the last relation has been obtained by Bernstein [1, p. 50].

2° In the design of electric filters the polynomial f_n is costrained to be an even function of frequency and if, in addition, a monotonic magnitude response is required, the error integral has the form

$$\text{(4.14)}\qquad I=\int_0^1 (1-x^2)^{p-q}x^{2q-1}(a_0+a_1x^2+\cdots+a_nx^{2n})^2\,dx\qquad (p-q>-1,\ q>0).$$

Again the minimizing function $x\mapsto f_n(x^2)$ and the minimum value of the integral are called for subject to the condition $f_n(1)=1$.

Since

$$\text{(4.15)}\qquad \int_0^1 (1-y)^{p-q}y^{q-1}G_j(p,q,y)\,G_k(p,q,y)\,dy=0\qquad (j,k=0,1,2,\ldots;\ j\neq k)$$

where G_n is the shifted Jacobi polynomial defined by

$$\text{(4.16)}\qquad G_n(p,q,y)=\frac{\Gamma(n+q)}{\Gamma(2n+p)}\sum_{r=0}^n(-1)^r\binom{n}{r}\frac{\Gamma(2n+p-r)}{\Gamma(n+q-r)}y^{n-r},$$

we find, by substituting $y=x^2$ in (4.16),

$$(4.17)\quad 2\int_0^1 (1-x^2)^{p-q}x^{2q-1}G_j(p,q,x^2)\,G_k(p,q,x^2)\,\mathrm{d}x=0\,(j,k=0,1,2,\ldots;j\neq k).$$

Also, from Theorem 2, the polynomial $x\mapsto R_n(x^2)$ that minimizes (4.17) is associated with the weight function $x\mapsto(1-x^2)\,w(x)=(1-x^2)^{p-q+1}x^{2q-1}$ so that

$$(4.18)\qquad Q_n(x^2)=G_n(p,q,x^2),\quad R_n(x^2)=G_n(p+1,q,x^2).$$

Since $k_n^Q=1$, $k_n^R=1$,

$$(4.19)\qquad G_n(p,q,1)=\frac{\Gamma(n+p)\,\Gamma(n+p-q+1)}{\Gamma(2n+p)\,\Gamma(p-q+1)},$$

$$(4.20)\qquad h_n=\frac{1}{2}\,\frac{n!\,\Gamma(n+q)\,\Gamma(n+p)\,\Gamma(n+p-q+1)}{(2n+p)\,\Gamma(2n+p)^2}$$

we get from (4.3) and (4.14)

$$(4.21)\qquad \min\int_0^1(1-x^2)^{p-q}x^{2q-1}(a_0+a_1x^2+\cdots+a_nx^{2n})^2\,\mathrm{d}x=\frac{n!\,\Gamma(n+q)\,\Gamma(p-q+1)\,\Gamma(p-q+2)}{2\,\Gamma(n+p+1)\,\Gamma(n+p-q+2)}$$

and from (4.4)

$$(4.22)\qquad \sum_{i=0}^{n}\frac{n!\,\Gamma(n+q)\,\Gamma(2i+p+1)}{i!\,\Gamma(i+q)\,\Gamma(2n+p+1)}G_i(p,q,x)=G_n(p+1,q,x).$$

If, instead of $f_n(1)=1$, the condition $f_n(0)=1$ is imposed, we get

$$(4.23)\qquad \min\int_0^1(1-x^2)^{p-q}x^{2q-1}(1+a_1x^2+\cdots+a_nx^{2n})^2\,\mathrm{d}x=\frac{n!\,\Gamma(n+p-q+1)\,\Gamma(q)\,\Gamma(q+1)}{2\,\Gamma(n+p+1)\,\Gamma(n+q+1)}$$

and

$$(4.24)\quad \sum_{i=0}^{n}(-1)^{n+i}\frac{n!\,\Gamma(n+p-q+1)\,\Gamma(2i+p+1)}{i!\,\Gamma(i+p-q+1)\,\Gamma(2n+p+1)}G_i(p,q,x)=G_n(p+1,q+1,x).$$

3° Theorem 1 and 2 still hold if one of the integration limits is infinite. For example, with $w(x)=x^\alpha e^{-x}$, $a=0$, $b=+\infty$, we have from (4.3)

$$(4.25)\qquad \min\int_0^{+\infty}x^\alpha e^{-x}(1+a_1x+\cdots+a_nx^n)^2\,\mathrm{d}x=\frac{\Gamma(\alpha+1)}{\binom{n+\alpha+1}{n}}.$$

For $\alpha=0$ this result, which find applications in the design of pulse forming networks, was proved Mordell [2].

Also from (4.4) the well known formula for the LAGUERRE polynomials is obtained

$$\sum_{i=0}^{n} L_i^{\alpha}(x) = L_n^{\alpha+1}(x). \tag{4.26}$$

ACKNOWLEDGMENTS. The authors wish to acknowledge the Research Fund of SR Serbia for financial support of work of which that described forms a part.

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